Math for 3D/Games Programmers

9. Integrals

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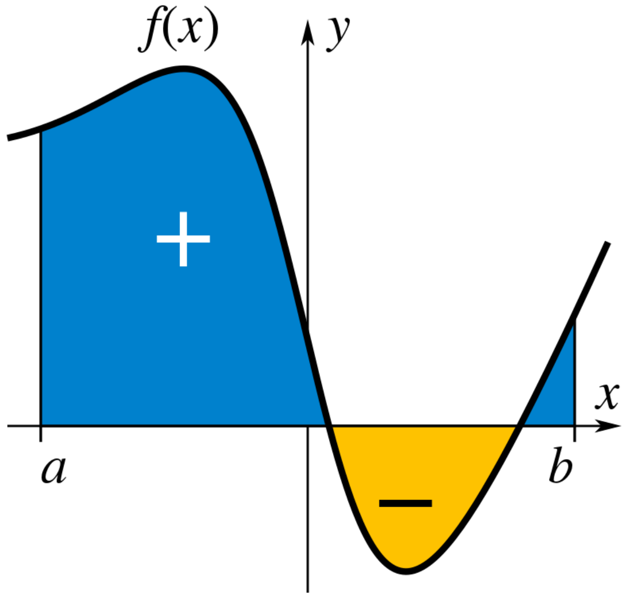
What is an Integral

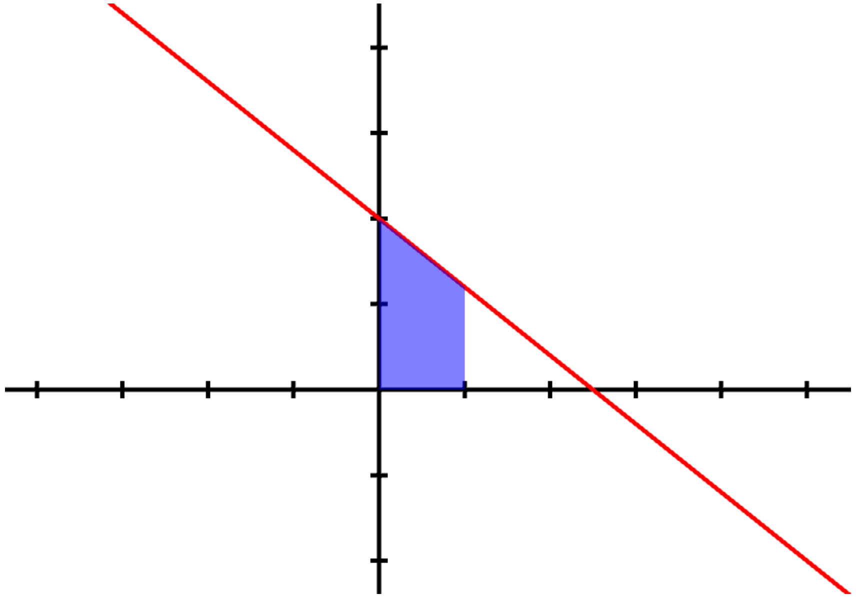
- **Integration** is the inverse operation to differentiation (derivative calculation)
- If we have a function $f(x)$, and its derivative $f'(x)$, then calculating the integral of $f'(x)$ will give us $f(x)$ (almost)
- Graphically, integral is the area of a region between the function and the X axis:



https://en.wikipedia.org/wiki/File:Integral_example.svg Author KSmrq

What is an Integral

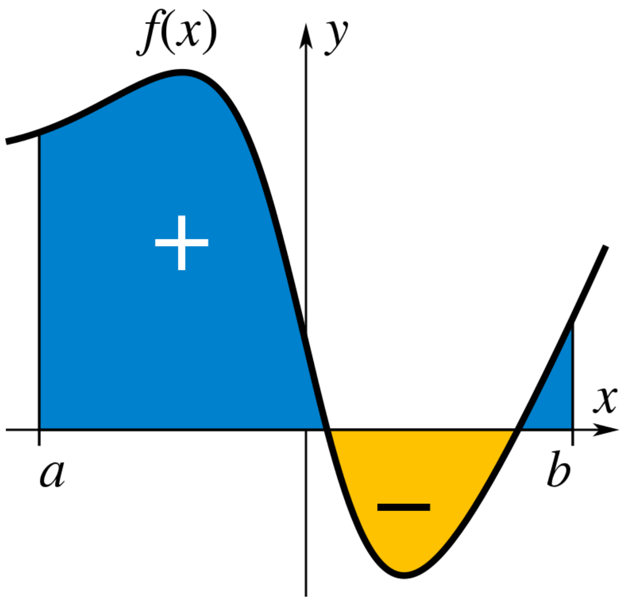
- Calculating the area between the X axis and, for example, a straight line is easy:



- This area is also an integral

What is an Integral

- However, how do we calculate the area between the X axis and any function?



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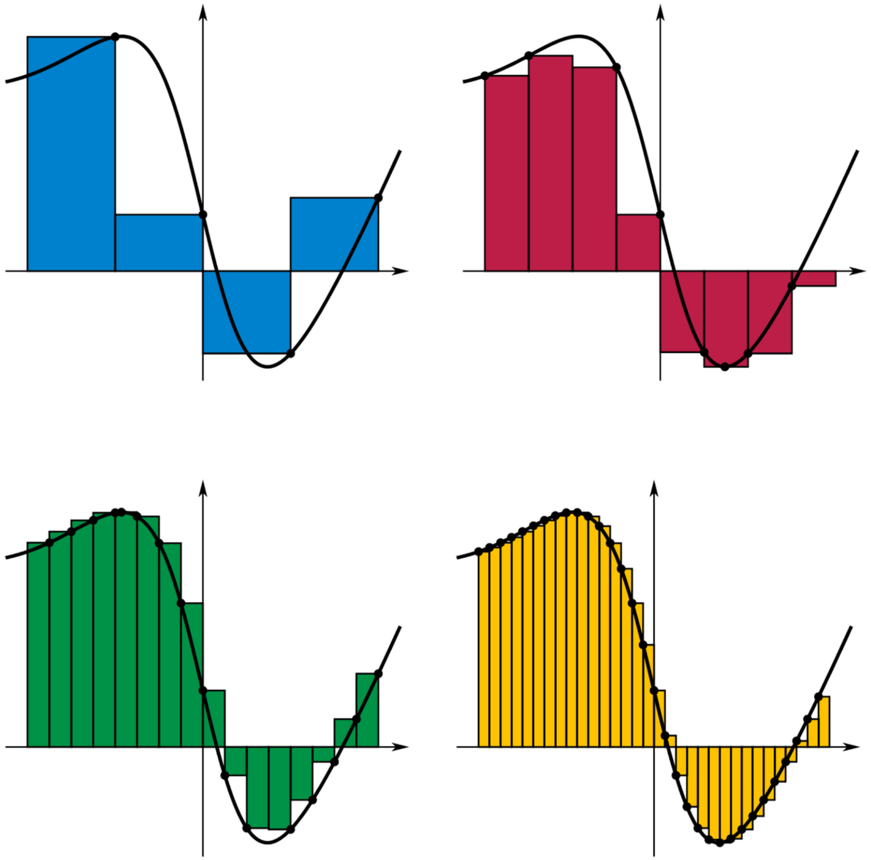
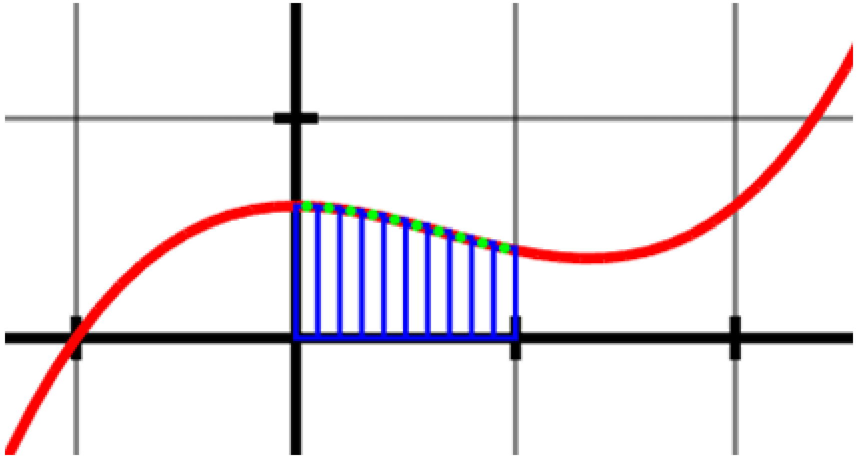


https://en.wikipedia.org/wiki/File:Riemann_sum_convergence.png Author KSmrq

Numerical Calculation

- Knowing that an integral is just a sum of rectangles, we can write the integral of $f(x)$ on the interval $[a, b]$ as:

$$I_{a,b} \approx \sum_{i=1}^{n} f(x_i) * \frac{b-a}{n} \qquad x_i \in [a, b]$$



Numerical Calculation

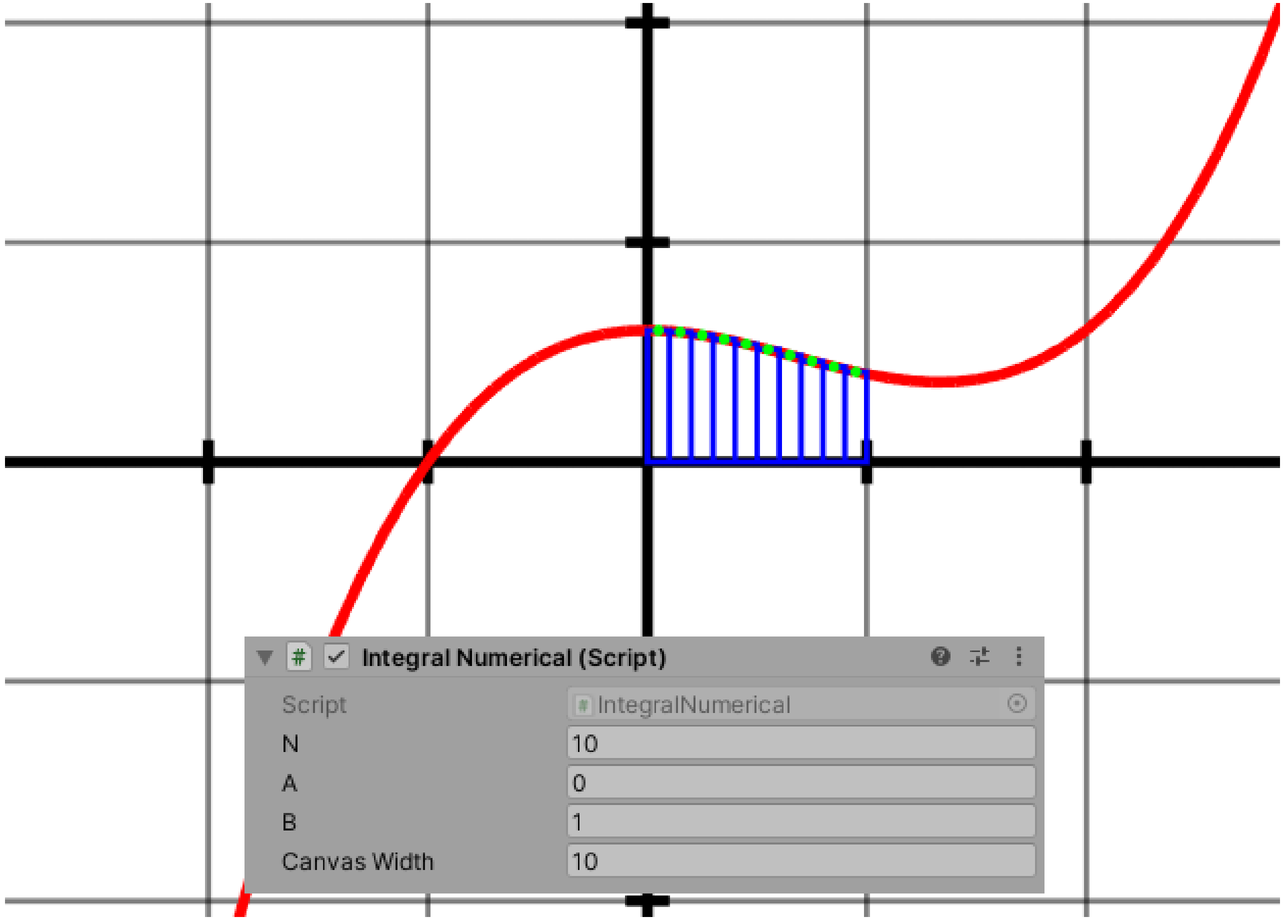
- Approximation:

$$I_{a,b} \approx \sum_{i=1}^{n} f(x_i) * \frac{b-a}{n} \qquad x_i \in [a, b]$$

- The exact value:

$$I_{a,b} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) * \frac{b-a}{n} \qquad x_i \in [a, b]$$

Numerical Calculation



Numerical Calculation

- Numerical integration allows us to de facto compute any integral; in the same unified way
- The accuracy of calculation depends on the number of divisions. Sometimes it may be necessary to perform a large number of calculations to obtain a satisfactory result
- We don’t necessarily need to use rectangles when dividing a function. There are other options as well (trapezoids for instance – Wikipedia)

Analytical Calculation

- The integration symbol is $\int$
- The integral of function $f(x)$:

$$I = F(x) = \int f(x)dx$$

Analytical Calculation

- As we said before, integration is the inverse of differentiation
- Example:

$$f(x) = 4x^2 - 3$$

$$F(x) = \int (4x^2 - 3)\,dx = 4\frac{x^3}{3} - 3x$$

- Because:

$$\left(4\frac{x^3}{3} - 3x\right)' = 4 * \frac{3x^2}{3} - 3 = 4x^2 - 3$$

Analytical Calculation

- Actually:

$$F(x) \;=\; \int (4x^2 - 3)\,dx \;=\; 4\frac{x^3}{3} - 3x + C$$

- Because:

$$\left(4\frac{x^3}{3} - 3x + C\right)' \;=\; 4 * \frac{3x^2}{3} - 3 \;=\; 4x^2 - 3$$

Analytical Calculation

- Calculating the integral of a polynomial function is easy:

$$\int (ax^n)dx \quad = \quad a\frac{x^{n+1}}{n+1} + C$$

$$\int (4x^2)\, dx \quad = \quad 4\frac{x^3}{3} + C$$

- Unfortunately, contrary to derivatives, analytical calculation of integrals is often significantly more involving and sometimes just plain impossible

Analytical Calculation

- By analytically integrating a function $f(x)$ we get a new function $F(x)$
- Where is the $[a, b]$ interval that indicates the integration range? And what do we do with the C constant

Analytical Calculation

- Let's say we have a function $f(x)$. We've also calculated its integral $F(x)$
- The area of the region between the function f and the X axis can be calculated as:

$$I_{a,b} = F(b) - F(a) = \int_a^b f(x)dx$$

- The C constants cancel out:

$$F(x) = 4\frac{x^3}{3} - 3x + C$$

$$F(b) - F(a) = \left(4\frac{b^3}{3} - 3b + C\right) - \left(4\frac{a^3}{3} - 3a + C\right)$$

Analytical Calculation

- In practice analytical calculation of such an integral looks like that:

$$\int_a^b (4x^2 - 3)dx \quad = \quad \left[4\frac{x^3}{3} - 3x\right]_a^b \quad = \quad \left(4\frac{b^3}{3} - 3b\right) - \left(4\frac{a^3}{3} - 3a\right)$$

- For example, for $a = 0$ and $b = 2$:

$$\left[4\frac{x^3}{3} - 3x\right]_a^b \quad = \quad \left(4\frac{b^3}{3} - 3b\right) - \left(4\frac{a^3}{3} - 3a\right) \quad = \quad \frac{14}{3} - 0 \quad = \quad \frac{14}{3}$$

Analytical Calculation

- When we are „just” calculating $F(x)$, we talk about **indefinite integral**:

$$\int f(x)dx$$

- When we are calculating a numerical value $F(b) - F(a)$, we talk about **definite integral**:

$$\int_a^b f(x)dx$$

- An indefinite integral $F(x)$ cannot give us a specific value due to the presence of the C constant. The $F(x)$ expression itself is also missing both ends of the integration interval

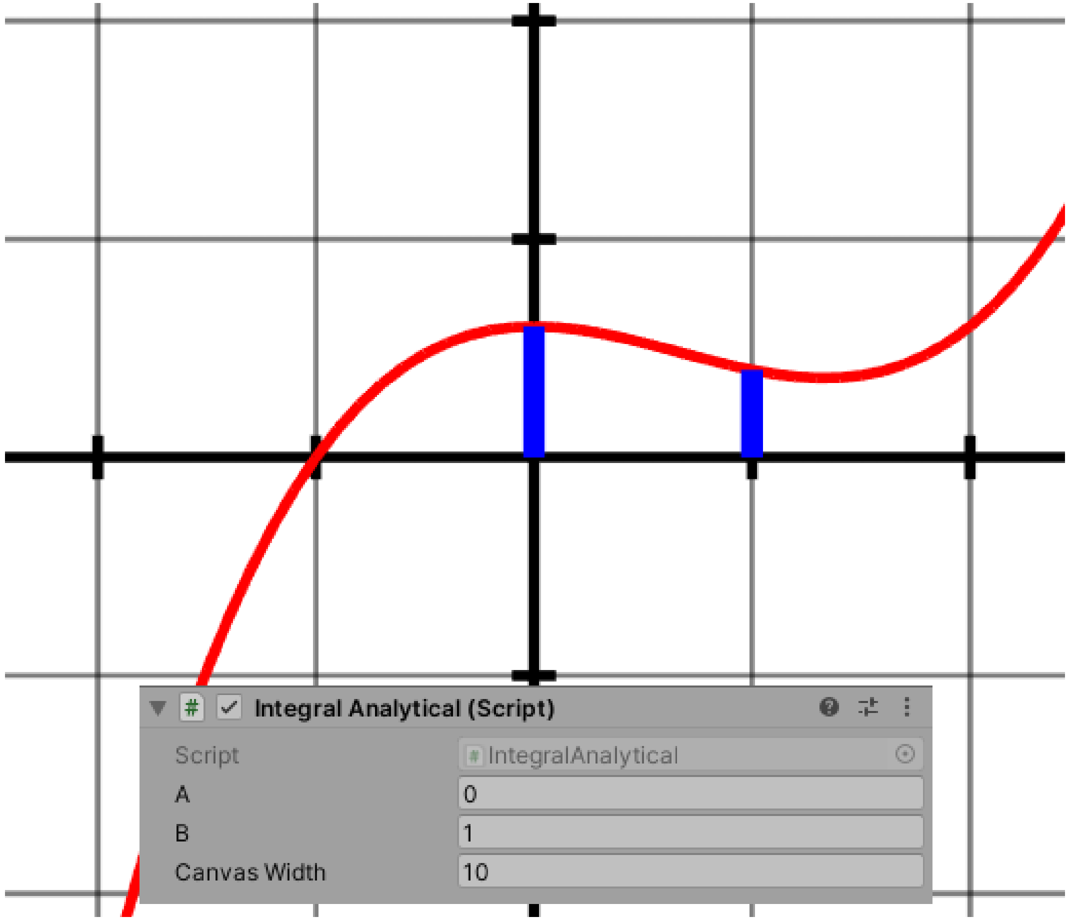
Analytical Calculation

- Note the similarity:

$$I_{a,b} = \int_a^b f(x)dx = F(b) - F(a)$$

$$I_{a,b} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) * \frac{b-a}{n} \qquad x_i \in [a, b]$$

Analytical Calculation



Analytical Calculation

- Wolfram

Analytical Calculation

- If we have the opportunity to calculate an integral analytically it will oftentimes be better than numerical integration
- The result of analytical integration is accurate and usually calculated quickly (in constant time)
- In practice however it turns out that analytical calculation of the integral of any given function is very difficult or even impossible. While there are general formulas that allow us to calculate the **derivative** of every function, in the case of **integrals** it is not the case
- As a result, in practice, the vast majority of integrals are calculated numerically

Motion Equation Integration

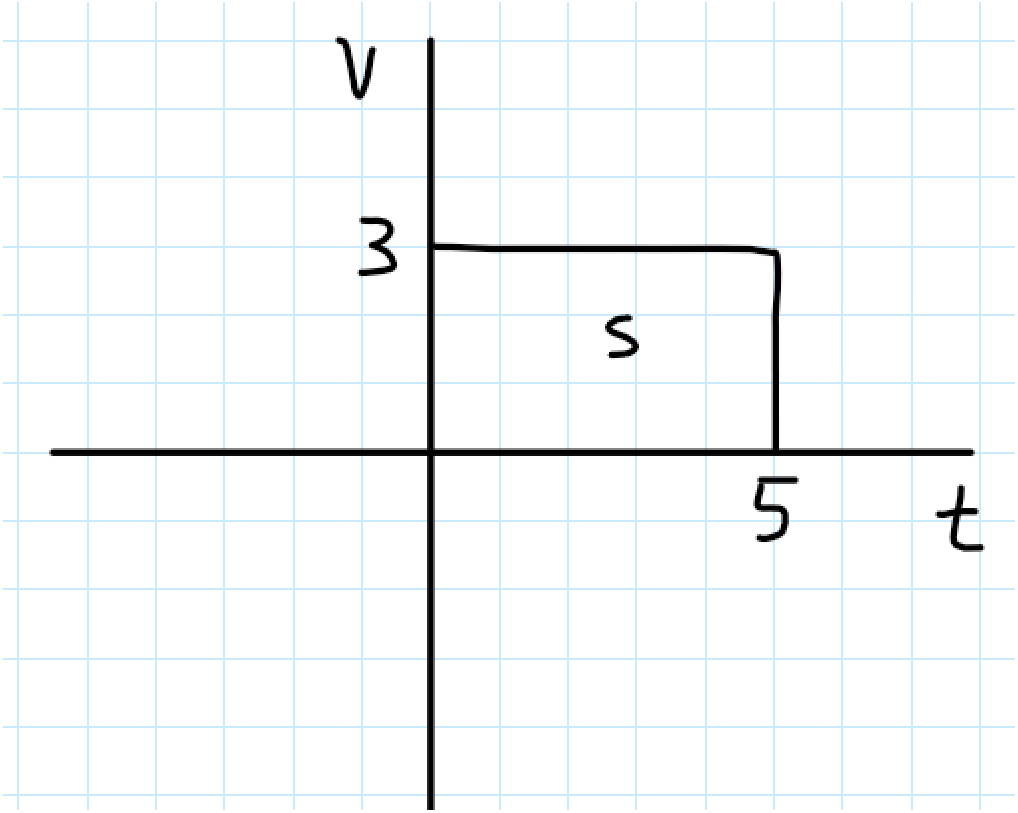
- Let's talk now about the relationship that integrals have with physics
- You likely remember a school formula for the distance traveled by a body in time t that moves at constant speed/velocity v:

$$s = vt$$

- A more „comprehensive" form:

$$s = s_0 + vt$$

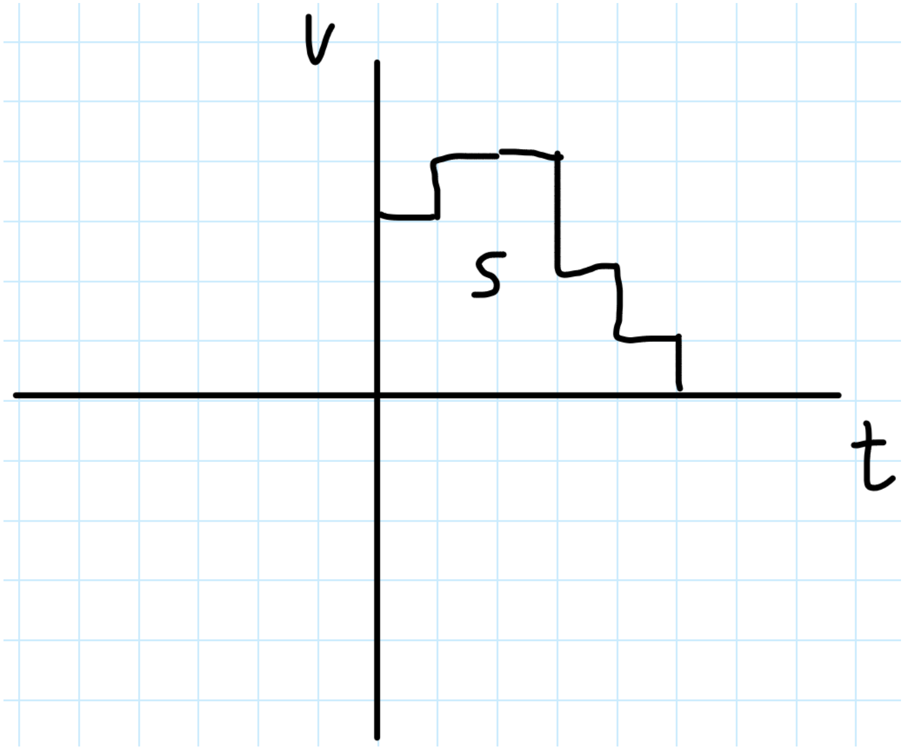
Motion Equation Integration



$t = 5$ $v = 3$ $s = vt = 15\,m$

Motion Equation Integration

- Speed usually is not constant



$$s = (3*1) + (4*1) + (4*1) + (2*1) + (1*1) = 14\,m$$

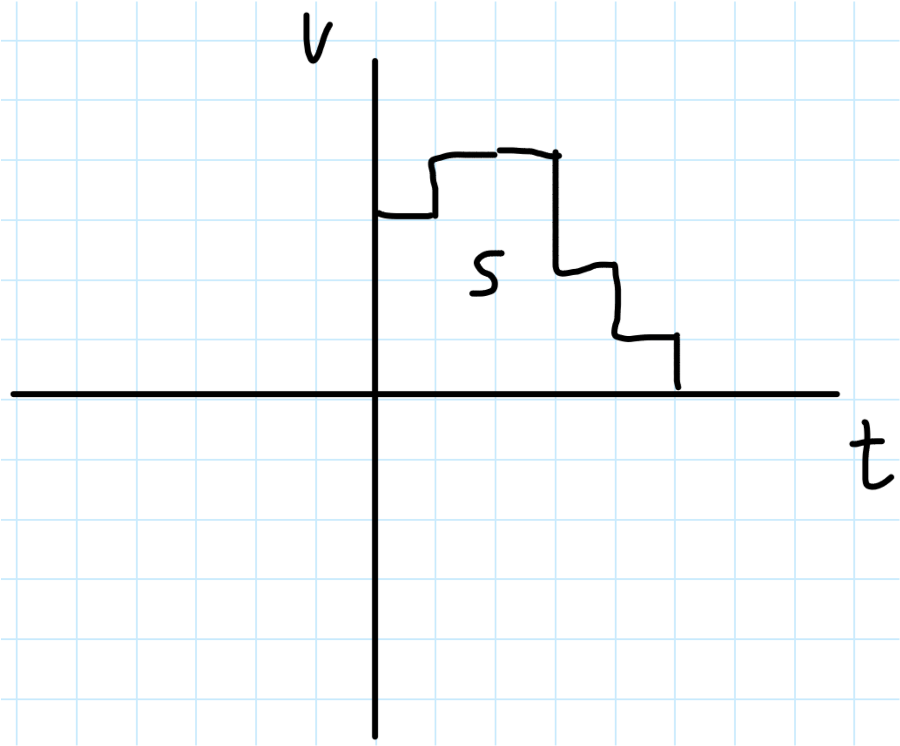
Motion Equation Integration

- Written more generically:

$$s = \sum_{i=1}^{5} v(t_i) * dt$$

$$dt = 1$$

$$t_i = i$$
$$t_i = i - 0.5$$



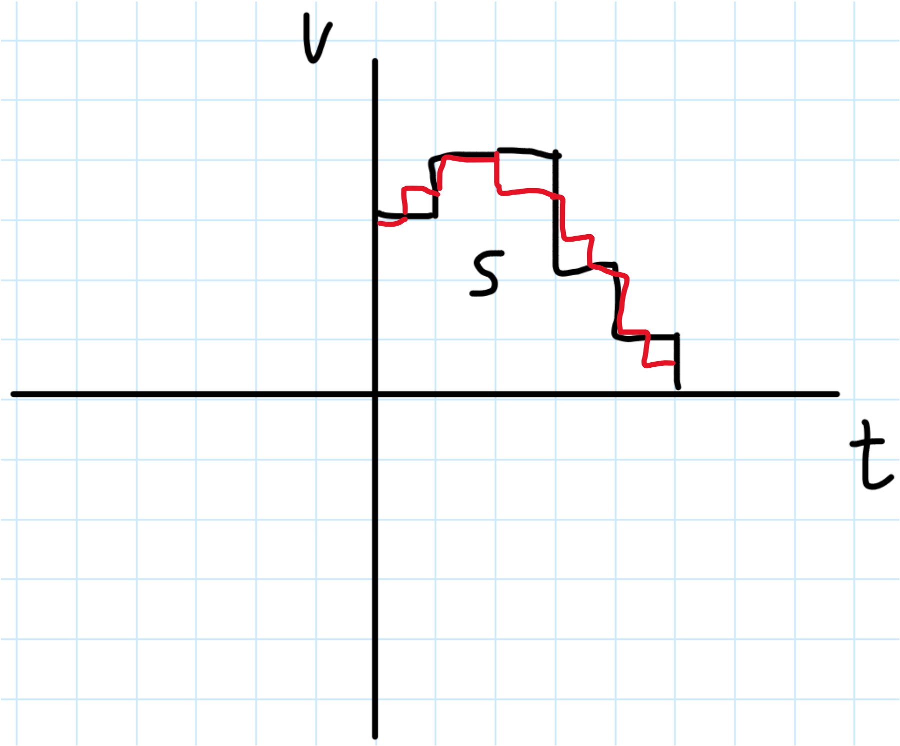
Motion Equation Integration

- 2x more „samples”:

$$s = \sum_{i=1}^{10} v(t_i) * dt$$

$$dt = 0.5$$

$$t_i = \frac{i}{2} - 0.25$$



Motion Equation Integration

- By calculating the distance traveled by a body we are actually integrating
- Function that we are integrating is the speed function $v(t)$ and the variable with respect to which we are integrating is t:

$$s(t) = \int v(t)dt$$

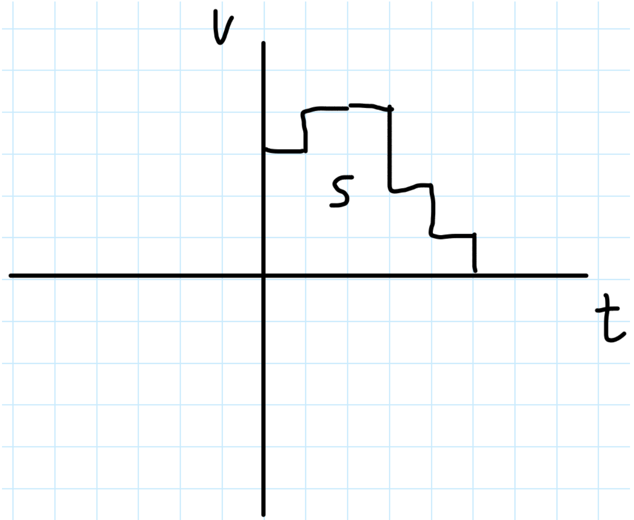
Motion Equation Integration

- Of course we usually calculate the distance on a specific time range $[t_1, t_2]$:

$$s = \int_{t_1}^{t_2} v(t)dt$$

- The above is now a definite integral which gives us a specific numeric value that is the distance traveled by a body starting at $t = t_1$ and ending at $t = t_2$
- Written using the sum operator:

$$s \approx \sum_{i=1}^{n} v(t_i) * \frac{t_2 - t_1}{n} \qquad t_i = t_1 + (i - 0.5)\frac{t_2 - t_1}{n}$$



Motion Equation Integration

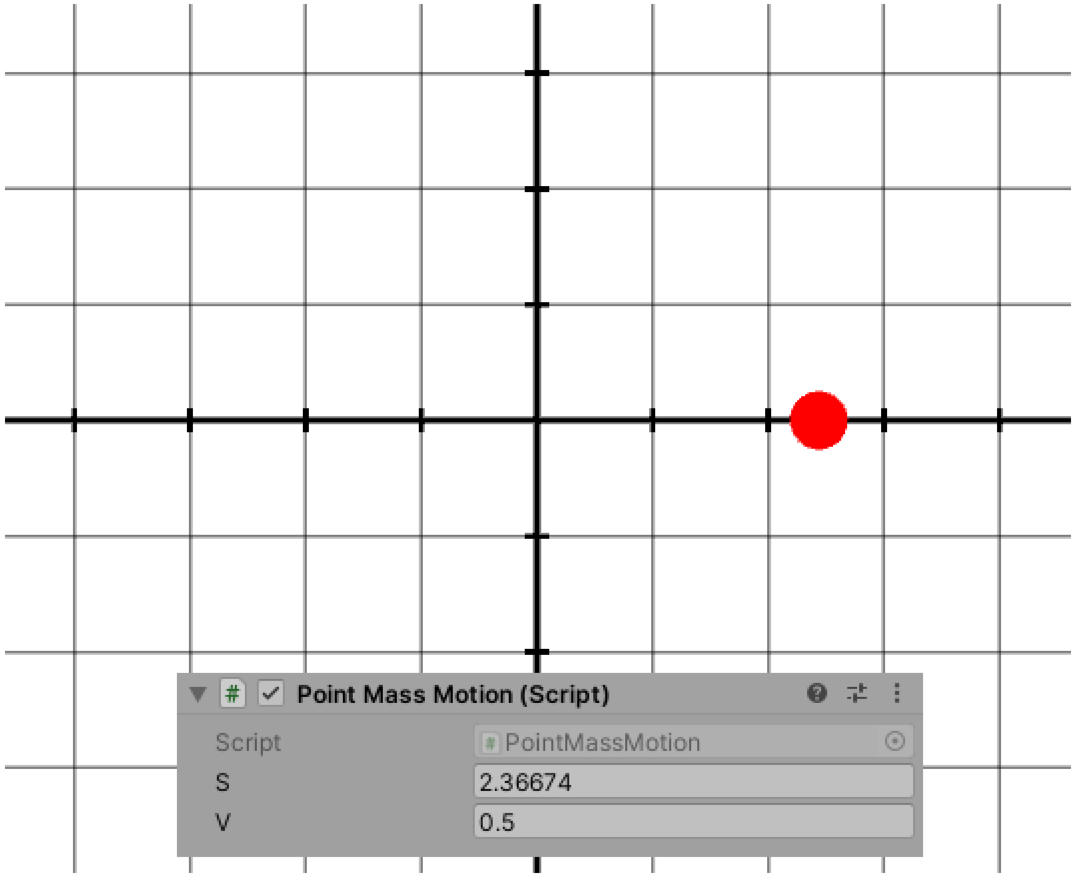
- If $v(t)$ is simple enough we can integrate analytically
- Let's integrate $v(t) = 3$:

$$s(t) = \int v(t)dt = \int 3dt = 3t + C$$

- And now let's calculate the distance traveled on the interval $[1, 4.5]$:

$$s(4.5) - s(1) = (3 * 4.5 + C) - (3 * 1 + C) = 13.5 - 3 = 10.5$$

Motion Equation Integration



Motion Equation Integration

- In the example program we calculated the integral numerically. In each frame (one `Update` call) we added – to the total distance traveled – the current speed multiplied by the time it took:

```
float s0 = s;
float dt = Time.deltaTime;

s = s0 + v*dt;
```

- This addition is integration
- Note that velocity here is actually a 1D vector (speed is its length)
- On that subject I highly recommend watching: https://www.youtube.com/watch?v=yGhfUcPjXuE

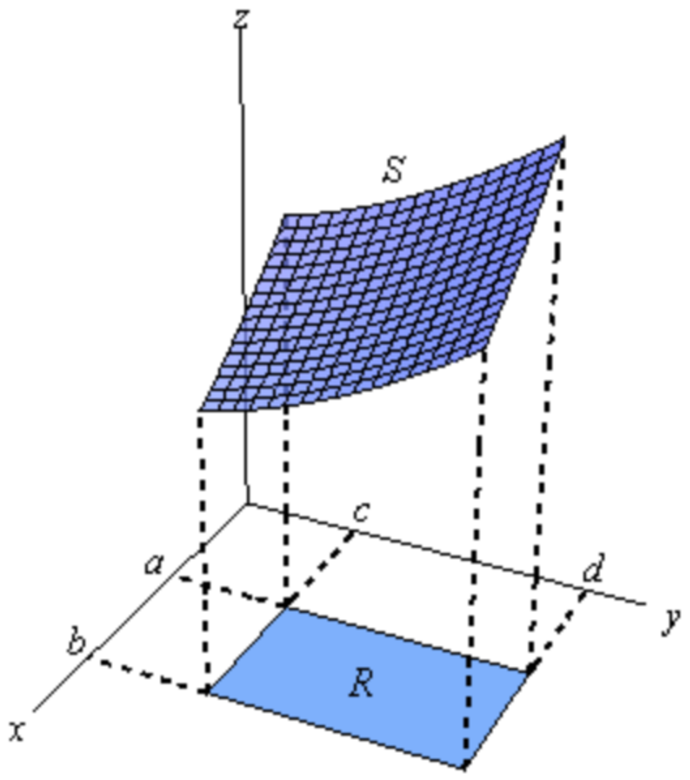
Double Integral

- **Double integral** is an integral that is calculated of a function that has two variables: $f(x, y)$
- It is similar in vein to derivatives where we could also calculate them with respect to more than one variable (partial derivatives)
- The graph of a function of two variables is called a **surface**:

$$z = f(x, y)$$

Double Integral

- An example surface S:

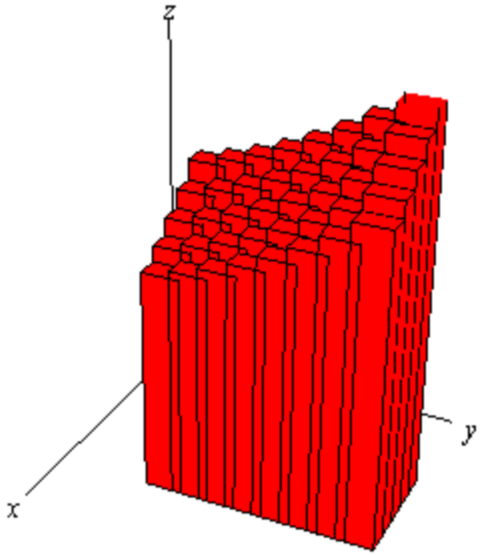


https://tutorial.math.lamar.edu/classes/calciii/DoubleIntegrals.aspx

Double Integral

- For a function of one variable the integral is equal to the area of the region between the graph of f and the X axis
- For a function of two variables the integral is equal to the volume between the surface of f and the XY plane
- The volume of each cuboid is:

$$V = f(x, y) * dx * dy$$



https://tutorial.math.lamar.edu/classes/calciii/DoubleIntegrals.aspx

Double Integral

- The above example demonstrated an integral calculated on the so-called **rectangular region**
- The region was a rectangle:

$$R = [a, b] \times [c, d]$$

- Integrals can be calculated on various regions (on a disk for instance). A rectangular region is easiest and deal with

Double Integral

- Let's now calculate the double integral of $f(x, y) = x^2 + y^2$ on region $R = [0, 2] \times [0, 1]$:

$$\iint_R f(x,y)dA \quad = \quad \iint_R f(x,y)dxdy \quad = \quad \int_0^1 \left(\int_0^2 f(x,y)dx \right) dy$$

$$dA = dx * dy$$

- Double integral calculation on such a region boils down to calculating two (definite) single integrals

Double Integral

$$\int_0^1 \left(\int_0^2 f(x,y)dx \right) dy \quad = \quad \int_0^1 \left(\int_0^2 (x^2 + y^2)dx \right) dy \quad =$$

$$\int_0^1 \left(\left[\frac{x^3}{3} + xy^2 \right]_0^2 \right) dy \quad = \quad \int_0^1 \left(\left(\frac{2^3}{3} + 2y^2 \right) - \left(\frac{0^3}{3} + 0y^2 \right) \right) dy \quad =$$

Double Integral

$$\int_0^1 \left(\left[\frac{x^3}{3} + xy^2 \right]_0^2 \right) dy \quad = \quad \int_0^1 \left(\left(\frac{2^3}{3} + 2y^2 \right) - \left(\frac{0^3}{3} + 0y^2 \right) \right) dy \quad =$$

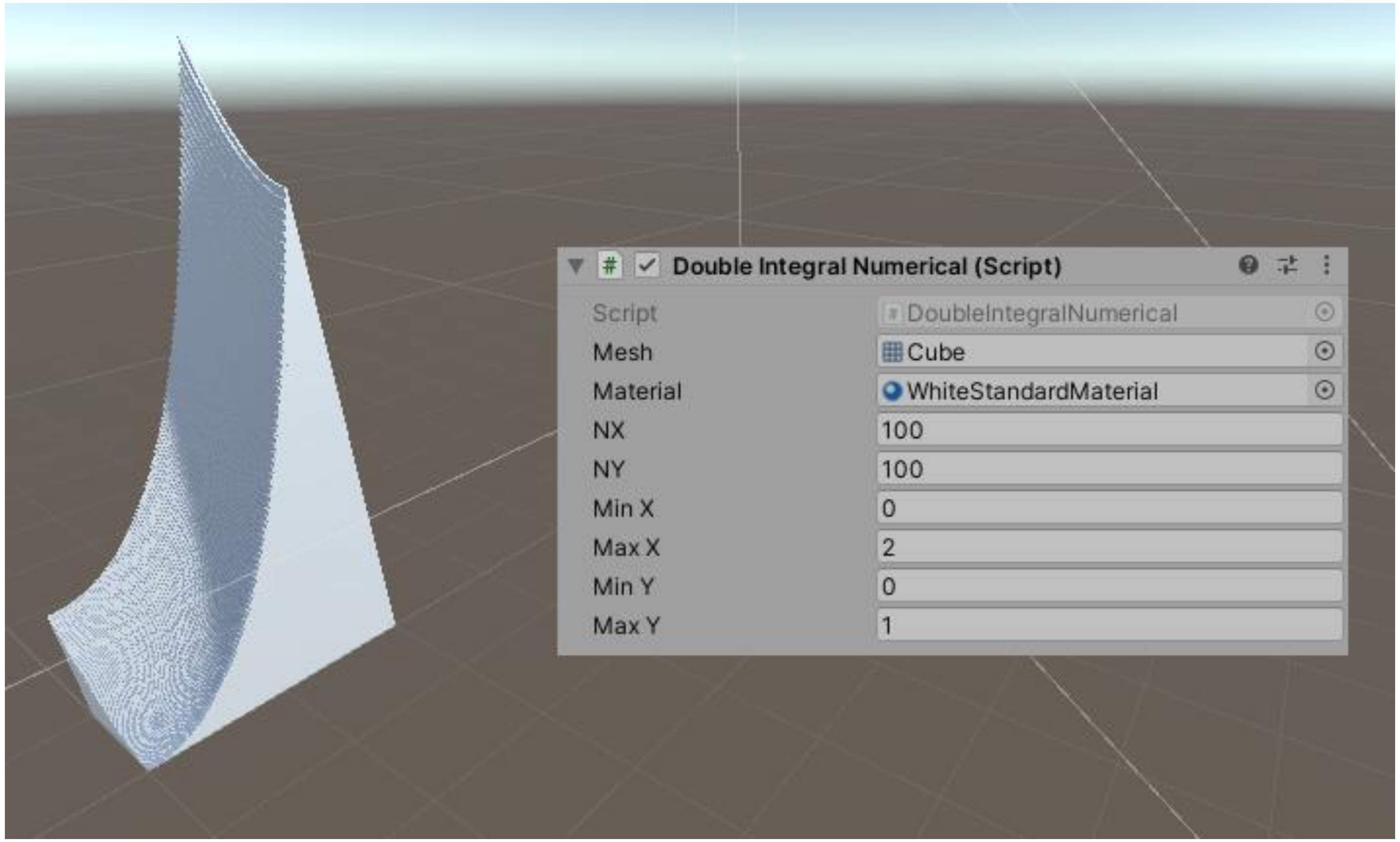
$$\int_0^1 \left(\frac{8}{3} + 2y^2 \right) dy \quad = \quad \left[\frac{8}{3} y + 2 \frac{y^3}{3} \right]_0^1 \quad = \quad \frac{8}{3} + \frac{2}{3} \quad = \quad 3\,\frac{1}{3}$$

Double Integral

- When integrating a rectangular region the order of integration does not matter:

$$\int_0^1 \left(\int_0^2 f(x,y)dx \right) dy \quad = \quad \int_0^2 \left(\int_0^1 f(x,y)dy \right) dx$$

Double Integral



Double Integral

- Double integrals often show up when calculating global illumination effects, where integration takes place on a sphere using spherical coordinates (variables θ and φ).
  A great source of information on that subject – containing many examples with accompanying source codes – is the book: Ray Tracing from the Ground Up
- A great generic source on integrals (and derivatives and other subjects) is: https://tutorial.math.lamar.edu/

Analytical Integration Example

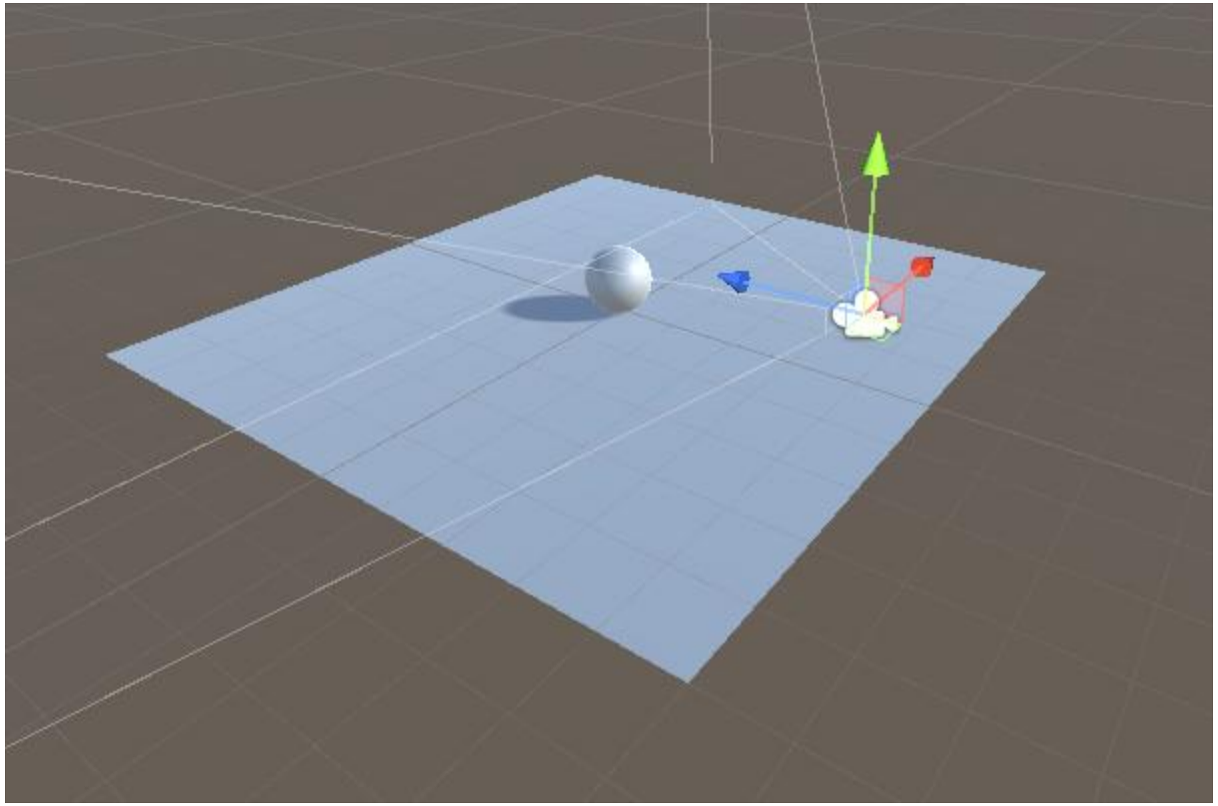
- Let's say we have a camera in a scene and we want it to move from point A to point B in some specific time T
- For this purpose we can use simple linear interpolation:

$$s(t) = \left(1 - \frac{t}{T}\right)A + \left(\frac{t}{T}\right)B$$

Analytical Integration Example

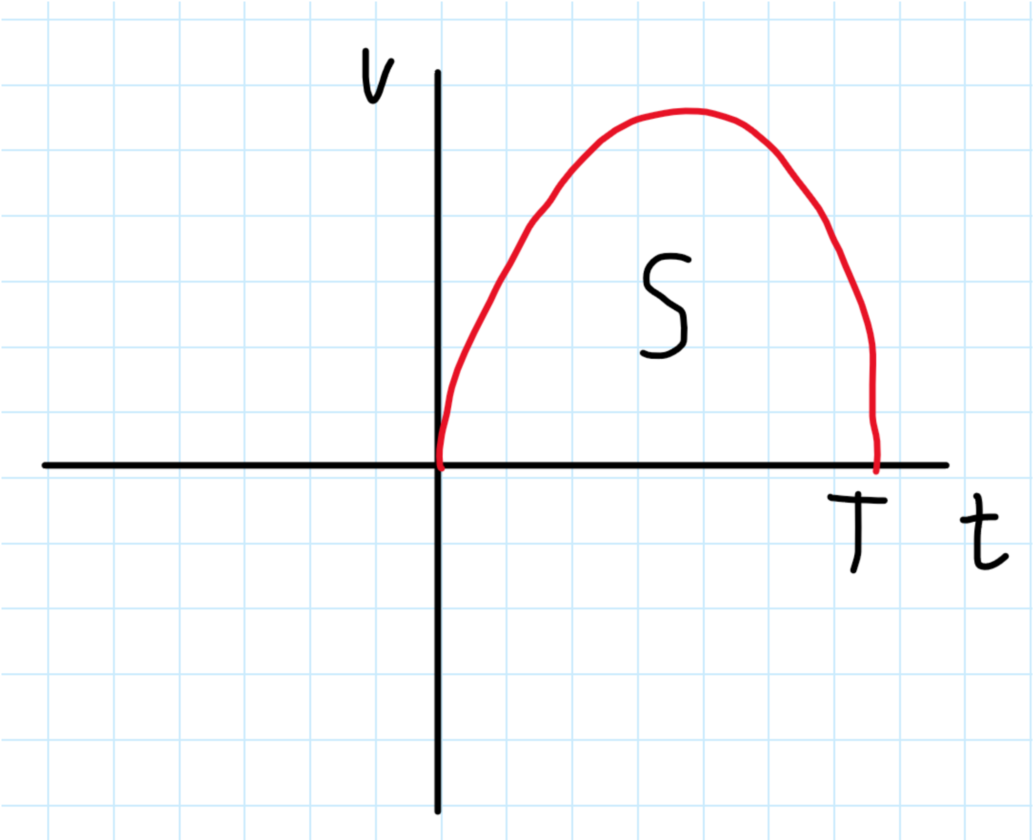
- This is not the best solution. Movement will be „stiff” – it will start suddenly at full speed and it will end suddenly. Furthermore speed will be the same throughout the whole distance
- Much more pleasant would be smooth movement – the camera starts at 0 speed, accelerates, at half the distance it will have reached the maximum speed, past which it starts to slow down. Movement ends at 0 speed

Analytical Integration Example



Analytical Integration Example

- We want the movement to look more or less like that:



Analytical Integration Example

- IMPORTANT OBSERVATION: We know upfront what the whole movement looks like!
- We assume that the speed will change in a parabola-like fashion
- The duration of the entire movement is T
- Since we know the start point and the end point we know the distance to travel S
- Our goal is to find the speed function $v(t)$, which on the time interval $[0, T]$ travels the distance S
- Knowing the speed function $v(t)$ we will be able to determine the distance function $s(t)$, which we can directly use to calculate the camera position

Analytical Integration Example

- The speed function $v(t)$ is said to be a parabola, so its formula is going to be of the form:

$$v(t) = \boldsymbol{a}t^2 + \boldsymbol{b}t + \boldsymbol{c}$$

- Our goal is to find coefficients a, b and c, which will allow us to determine the speed function $v(t)$ formula
- Since we have three unknowns, to find them we need three equations

Analytical Integration Example

- We know that at the beginning the speed is 0:

$$v(0) = 0$$

$$\boldsymbol{a}0^2 + \boldsymbol{b}0 + \boldsymbol{c} = 0$$

$$\boldsymbol{c} = 0$$

Analytical Integration Example

- We also know that at the end of the movement speed is 0:

$$v(T) = 0$$

$$\boldsymbol{a}T^2 + \boldsymbol{b}T + \boldsymbol{c} = 0$$

Analytical Integration Example

- We also know that the distance to travel is S
- In other words we know that the integral of the speed function (distance traveled) is S:

$$\int_0^T v(t)dt = S$$

$$\int_0^T (\boldsymbol{a}t^2 + \boldsymbol{b}t + \boldsymbol{c})dt = S$$

$$\int_0^T (\boldsymbol{a}t^2 + \boldsymbol{b}t + \boldsymbol{c})dt \quad = \quad \left[\frac{\boldsymbol{a}t^3}{3} + \frac{\boldsymbol{b}t^2}{2} + t\boldsymbol{c}\right]_0^T \quad = \quad \frac{\boldsymbol{a}T^3}{3} + \frac{\boldsymbol{b}T^2}{2} + T\boldsymbol{c} \quad = \quad S$$

Analytical Integration Example

- With that we have three equations:

$$\boldsymbol{c} = 0$$

$$\boldsymbol{a}T^2 + \boldsymbol{b}T + \boldsymbol{c} = 0$$

$$\frac{\boldsymbol{a}T^3}{3} + \frac{\boldsymbol{b}T^2}{2} + T\boldsymbol{c} = S$$

- Wolfram

Analytical Integration Example

- After solving this system we get:

$$a = -\frac{6S}{T^3}$$

$$b = \frac{6S}{T^2}$$

$$c = 0$$

$$v(t) \;=\; \boldsymbol{a}t^2 + \boldsymbol{b}t + \boldsymbol{c} \;=\; -\frac{6S}{T^3}t^2 + \frac{6S}{T^2}t$$

Analytical Integration Example

- Knowing $v(t)$ we can come up with the formula for the distance traveled $s(t)$:

$$s(t) \;=\; \int v(t)dt$$

$$s(t) \;=\; \int \left(-\frac{6S}{T^3}t^2 + \frac{6S}{T^2}t\right)dt$$

$$s(t) \;=\; \left(-\frac{6S}{T^3} * \frac{t^3}{3}\right) + \left(\frac{6S}{T^2} * \frac{t^2}{2}\right) + C$$

Analytical Integration Example

- The formula for the distance traveled includes the C constant. No worries, since in reality to calculate the distance we have to calculate the difference between two points of $s(t)$
- For example, it is the calculation of:

$$s(t) - s(0)$$

  that gives us the distance which a body has traveled in time t

- During calculation of that difference the C constants cancel out

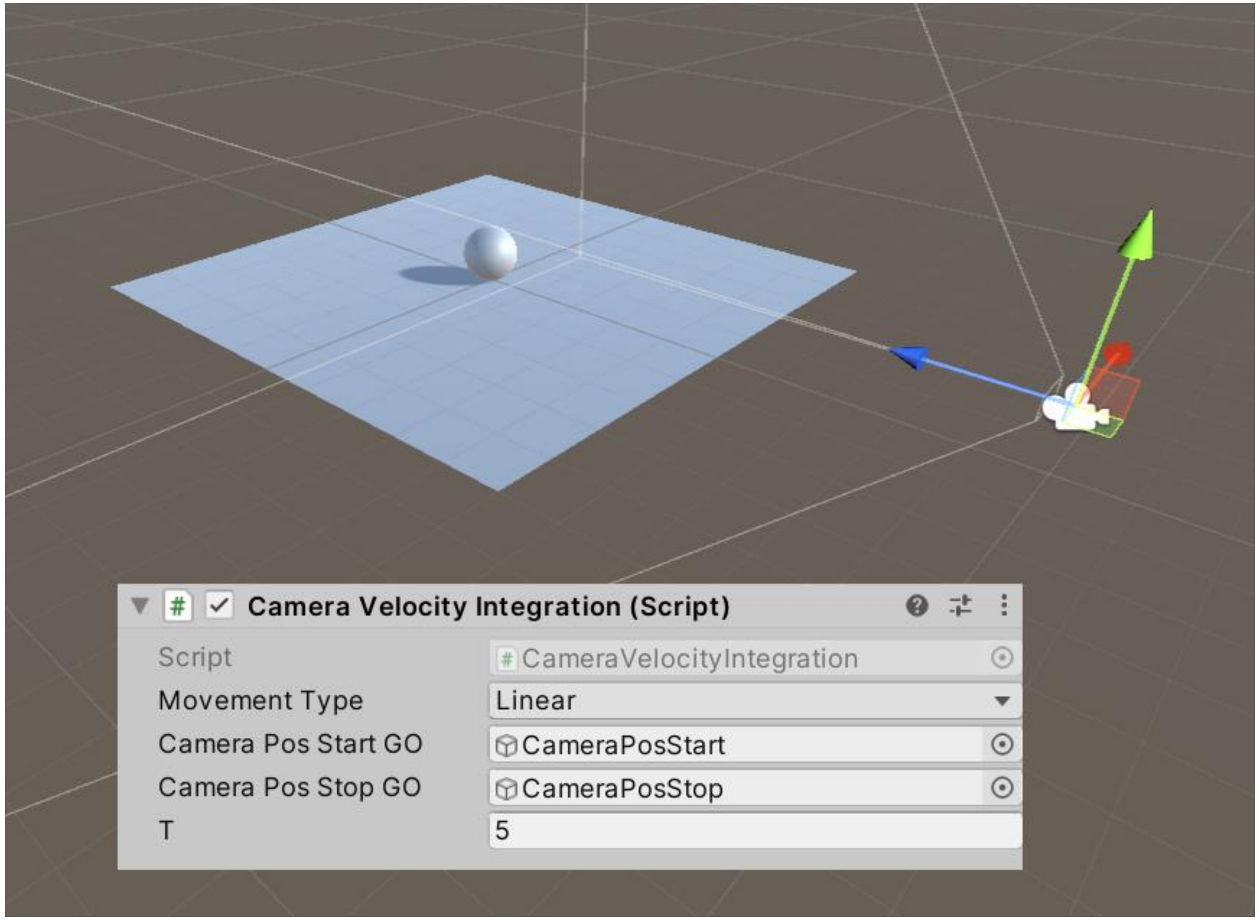
Analytical Integration Example

- The expression/formula:

$$s(t) - s(0) \;=\; \left(-\frac{6S}{T^3} * \frac{t^3}{3}\right) + \left(\frac{6S}{T^2} * \frac{t^2}{2}\right)$$

gives us the distance traveled in time t (assuming that the movement started at $t = 0$)

Analytical Integration Example



Analytical Integration Example

- Note that in this example we calculated an integral for the entire distance to travel, in a single formula. We could do it because we knew upfront how the whole movement is supposed to look like (parabola)
- When we talked about integration of the motion equation, a single step of integration was performed each time in the `Update` function. The value of $v(t)$ was „learned on the go”, as we changed it via the Unity’s inspector

Analytical Integration Example

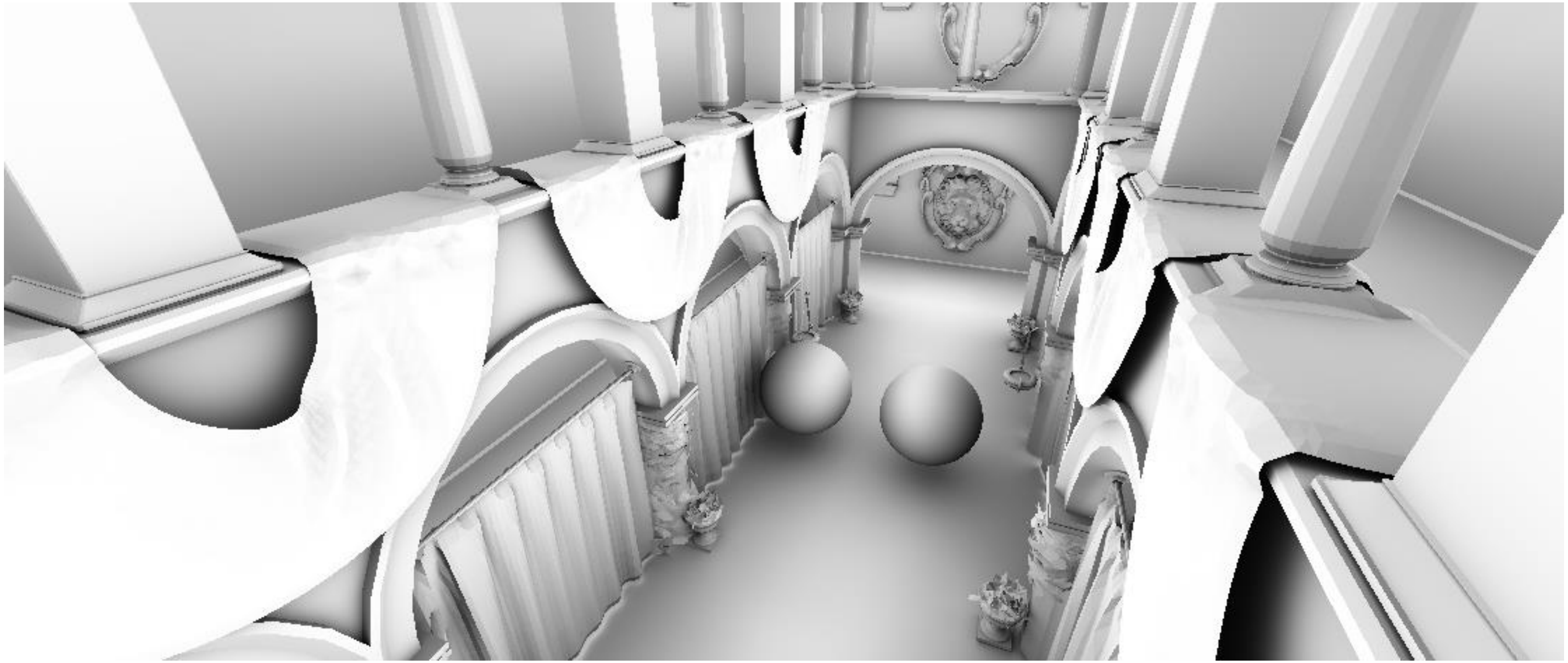
- In practice, we quite rarely deal with an analytic function described with a formula. Because of that the opportunity of analytical integration rarely arises
- Theoretically, every function/data can be represented with some analytic function. In practice however, deriving such a function will be very problematic and the function itself will have a complicated formula
- Whenever possible, it is worth considering using analytical integration

Numerical Integration Example

- In the context of analytical integrals the most important thing is understanding what they represent
- Various technical concepts are oftentimes presented using the integral apparatus, even though it might not be at all possible to calculate the integral directly
- Sometimes a problem can be conveniently formulated as an integral and then transformed/simplified as much as possible. However, for the final calculation, it will usually be necessary to perform numerical integration – i.e. conversion of the integral into a (finite) sum

Numerical Integration Example

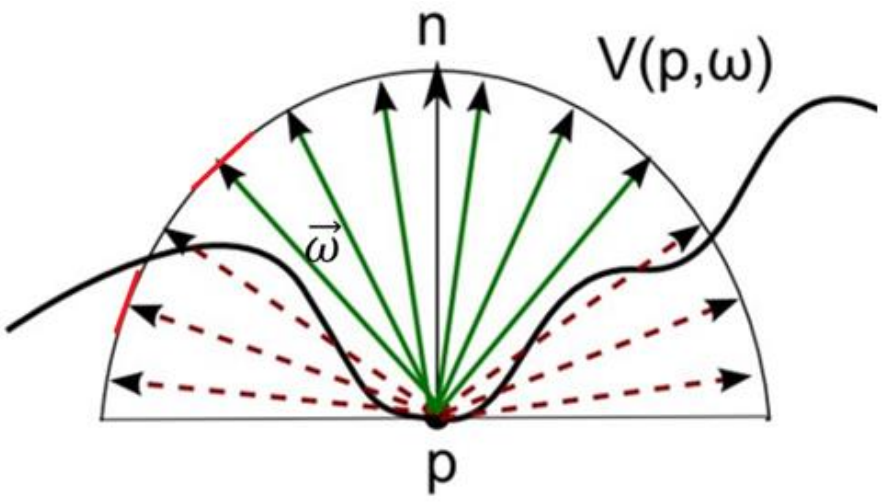
- As an example we will see how to solve the problem of ambient occlusion:



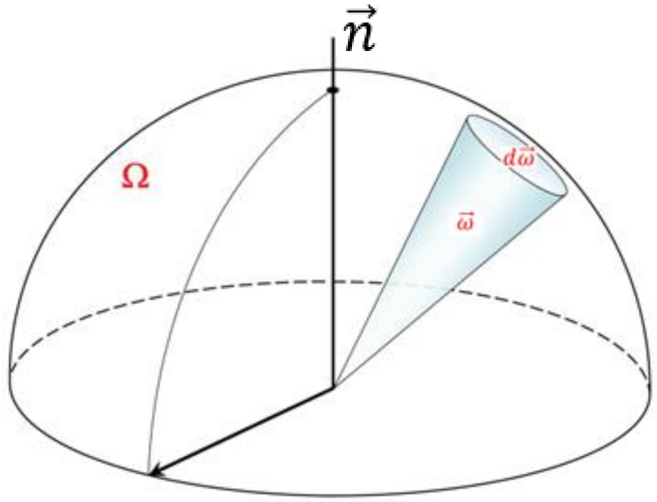
Numerical Integration Example

- The ambient occlusion formula at point p with normal vector $\vec{n}$:

$$AO(p,\vec{n}) \approx \frac{1}{\pi}\sum_{i=1}^{N} V(p,\vec{\omega}_i)(\vec{n} \circ \vec{\omega}_i)\frac{2\pi}{N}$$



https://nbertoa.files.wordpress.com/2017/02/ao_integral.png?w=756



https://seos-project.eu/laser-rs/laser-rs-c03-s02-p02.html

Numerical Integration Example

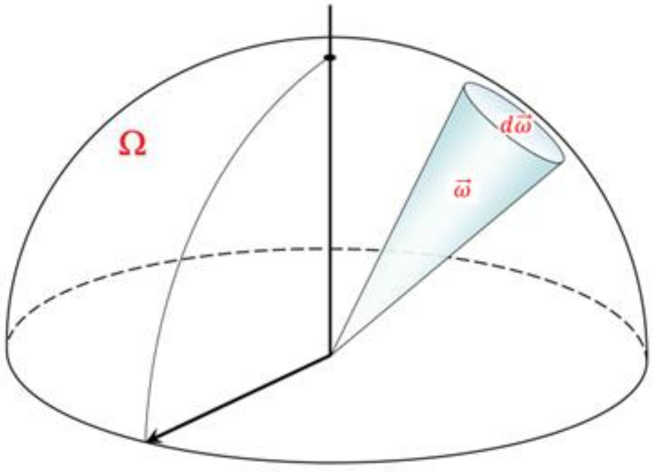
- The formula:

$$AO(p,\vec{n}) \approx \frac{1}{\pi}\sum_{i=1}^{N} V(p,\vec{\omega}_i)(\vec{n} \circ \vec{\omega}_i)\frac{2\pi}{N}$$

- Written as an integral:

$$AO(p,\vec{n}) = \frac{1}{\pi}\int_{\Omega} V(p,\vec{\omega})(\vec{n} \circ \vec{\omega})d\vec{\omega}$$

- Wikipedia



https://seos-project.eu/laser-rs/laser-rs-c03-s02-p02.html

Numerical Integration Example

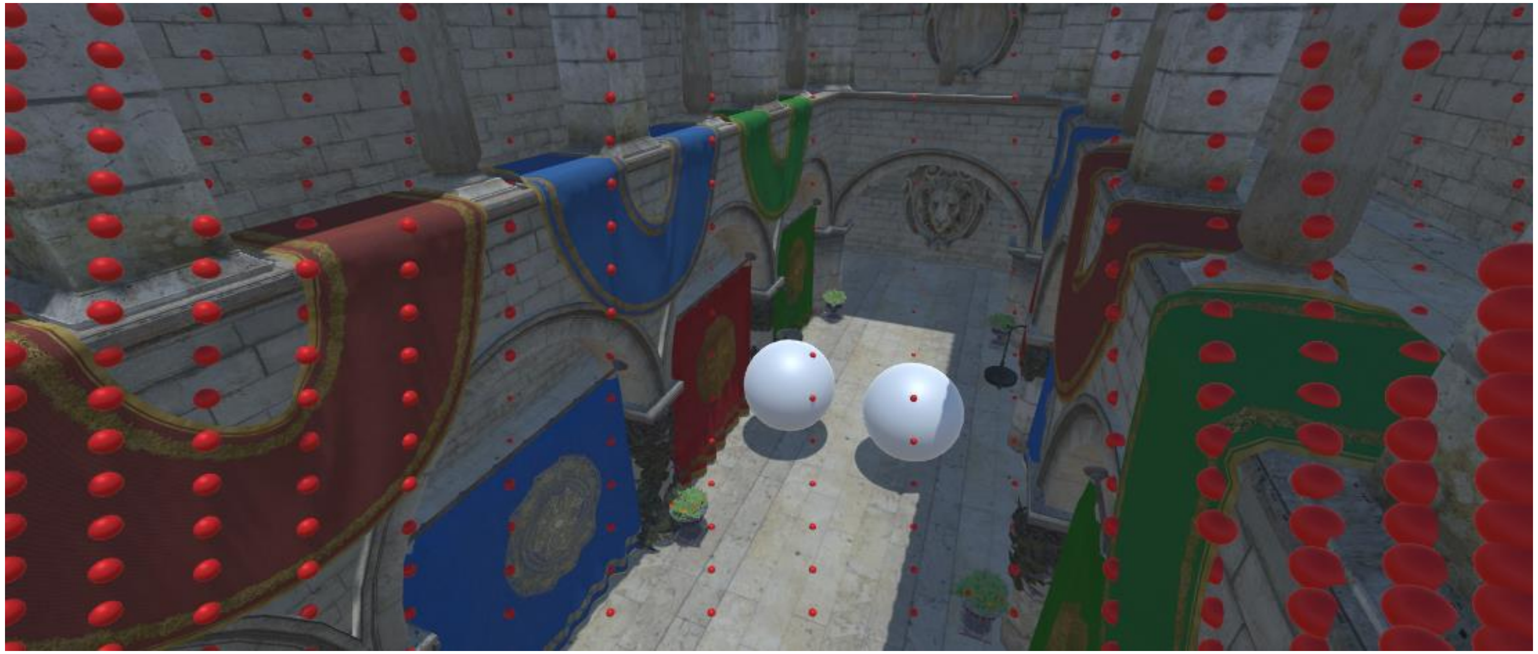
- The term $\frac{1}{\pi}$ is a normalization term needed in the AO formula because:

$$\int_{\Omega} (\vec{n} \circ \vec{\omega}) d\vec{\omega} = \pi$$

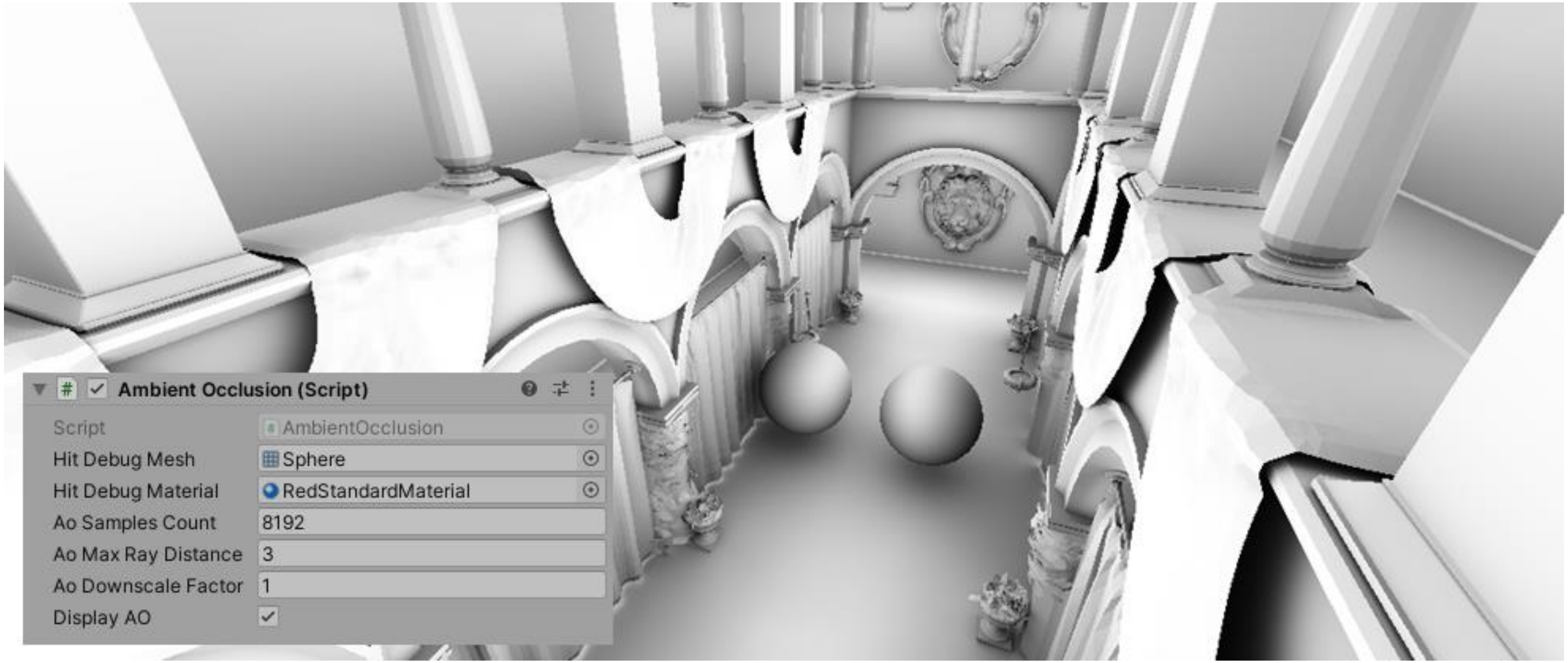
- Thanks to it, when V returns 1 for all directions (no occluding geometry), AO will return 1 and not π

Numerical Integration Example

- To calculate ambient occlusion we first need to find – for all points visible on the screen – their coordinates p and normal vectors $\vec{n}$ (all in world space)
- We can do that by „shooting" rays starting at the near plane of the camera towards the far plane of the camera



Numerical Integration Example



Numerical Integration Example

- Analytical integration can be used to transform a problem but the final result will usually need to be calculated numerically
- The Alchemy Screen-Space Ambient Obscurance Algorithm

$$A_r^*(C, \hat{n}) = \frac{1}{\pi} \int_\Omega V(C, C + r\hat{\omega})\, \hat{\omega} \cdot \hat{n}\, d\hat{\omega}$$

$$A = 1 - u \int_\Gamma \frac{\vec{v}(\hat{\omega}) \cdot \hat{n}}{\max\left(u^2, \vec{v}(\hat{\omega}) \cdot \vec{v}(\hat{\omega})\right)}\, d\hat{\omega}$$

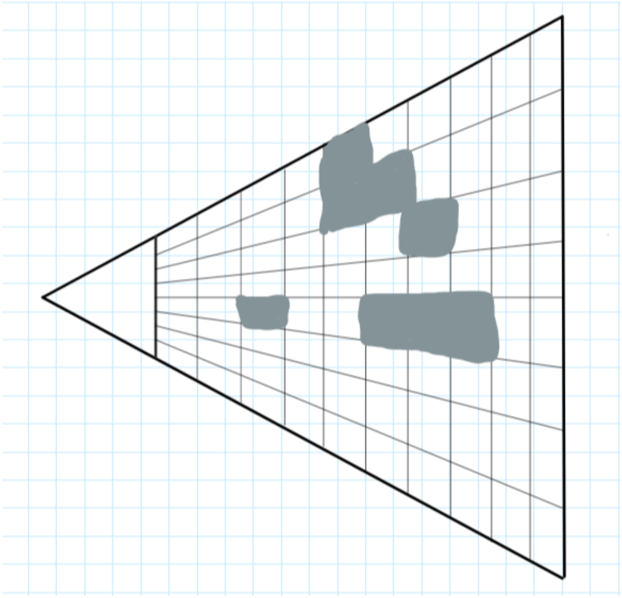
$$A \approx 1 - \frac{2\pi u}{s} \sum_{i=1}^{s} \frac{\max(0, \vec{v_i} \cdot \hat{n}) \cdot H(r - ||\vec{v_i}||)}{\max(u^2, \vec{v_i} \cdot \vec{v_i})}$$

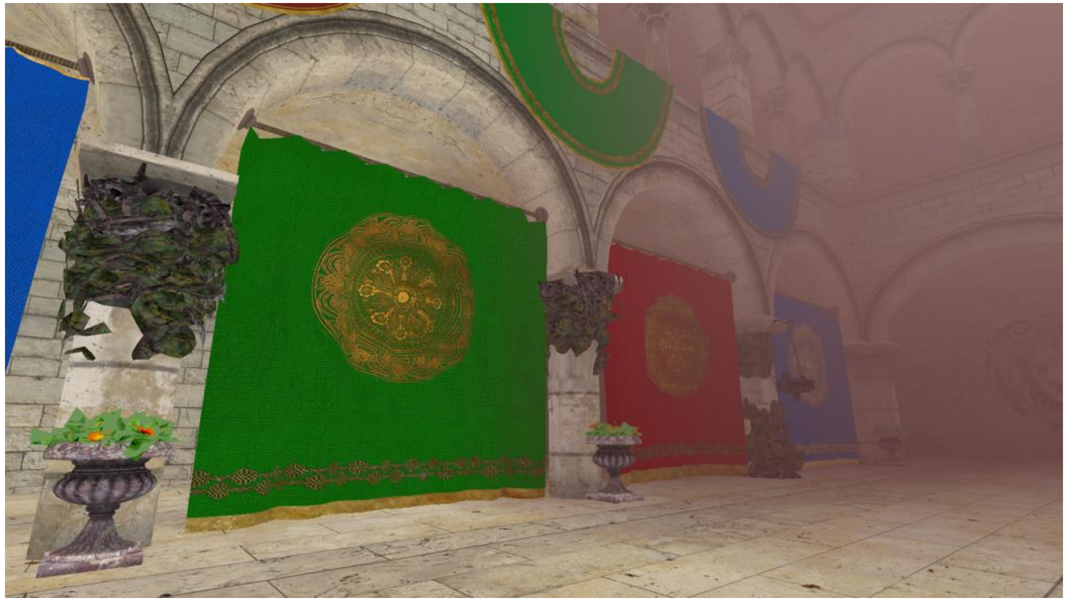
Numerical Integration Example

- Integration is very common when calculating global illumination effects (ambient occlusion is a component of that).
  A great source of information on that subject – containing many examples with accompanying source codes – is the book: Ray Tracing from the Ground Up

Numerical Integration Example

- Another good example is volumetric fog:





- Example code and the same thing implemented in Unreal Engine

Exercises

- In the PointMassMotion program implement movement in 2D (slide 30)
- In the PointMassMotion program include acceleration a into calculations (slide 30)
- In the CameraVelocityIntegration program we derived an analytic formula for the traveled distance s in time t, and we used it to calculate the camera's position.
  We also derived the velocity formula v.
  Change the way the camera's position is calculated in that program, so that it does not use the distance formula s directly, but instead use numerical integration by sampling function v (slide 56)