

Math for 3D/Games Programmers

2. Complex Numbers

DISCLAIMER

- Complex numbers are not particularly common in 3D/games programming, so we will only skim over them
- The main reason we are touching on this subject is because complex numbers are foundation for **quaternions**, which in turn are widely used in 3D/games programming, and which will be discussed in chapter 7
- Complex numbers will also pop up in chapter 8 while solving one particular equation
- Moreover, complex numbers are important in other domains (e.g. Fourier transform) and it is worthwhile to have some basic understanding about them

Table of Contents

- Definition
- Operations
- Trigonometric Form
- Exponential Form

Definition

- **Complex number** is a number of the following form:

$$z = a + bi \quad a, b \in R$$

i – imaginary unit, such that:

$$i = \sqrt{-1} \quad \text{or} \quad i^2 = -1$$

- Complex number is therefore represented by a pair of real numbers (a, b) .
When $b = 0$ we are dealing with a regular real number a
- Above we're seeing the **algebraic form** of a complex number
- Just as on the number line we can mark real numbers, we can mark complex numbers in the 2D coordinate system

Definition

- A few examples of complex numbers:

$$6 + 2i$$

$$-4 - \sqrt{3}i$$

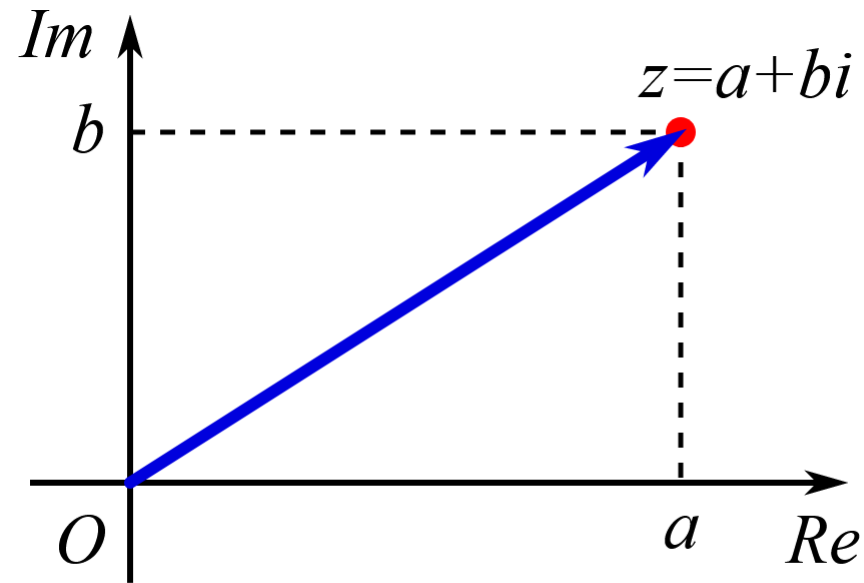
$$-7i$$

$$5$$

$$5 - 9\sqrt{-1}$$

$$-3 + 4j$$

Definition



https://en.wikipedia.org/wiki/File:A_plus_bi.svg Author IkamusumeFan

- Length of the blue arrow (distance of the z number from the origin) is the so called **modulus** („length”) of a complex number, which is calculated as:

$$|z| = r = \sqrt{a^2 + b^2}$$

Definition

- Note that since it is true that $i = \sqrt{-1}$, then in the complex numbers domain we can calculate roots of negative numbers, for example:

$$\sqrt{-4} = \sqrt{4} * \sqrt{-1} = 2\sqrt{-1} = 2i$$

- [Wolfram](#)
- The set of natural numbers \mathbb{N} is contained in the set of integer numbers \mathbb{Z} , while the set of real numbers \mathbb{R} is contained in the set of complex numbers \mathbb{C} :

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Definition

- It happens that when solving some equations, the problem may be formulated in real numbers \mathbb{R} , the solution may also be in real numbers \mathbb{R} , but transformations of equations along the way may involve complex numbers \mathbb{C}
- As an illustration let's solve some simple quadratic equation:

$$x^2 - 2x - 3 = 0$$

$$\Delta = b^2 - 4ac = 16 \quad \sqrt{\Delta} = 4$$

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = -1 \mid 3$$

We started and finished with integer numbers \mathbb{Z} , but along the way we had to deal with fractions (rational numbers \mathbb{Q})

Definition

- [The fascinating thing about complex numbers:](#)

Every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots

$x^7 - 5x^3 - 7 = 0$ will have 7 roots/solutions in \mathbb{C}

- A more trivial example:

$$x^2 + 1 = 0$$

does not have any solutions in \mathbb{R} .

In \mathbb{C} there are: $x_1 = i$ and $x_2 = -i$

$$x_1 = 0 + 1i \quad \text{and} \quad x_2 = 0 - 1i$$

Operations

- Many operations are defined on complex numbers, including operations that apply to real numbers, such as addition, multiplication, square root, etc.
- However, you cannot compare two complex numbers!
- For example, adding two complex numbers looks like this:

$$z_1 = a_1 + b_1 i$$

$$z_2 = a_2 + b_2 i$$

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$$

Operations

- Multiplying two complex numbers:

$$z_1 = a_1 + b_1i$$

$$z_2 = a_2 + b_2i$$

$$z_1z_2 = (a_1 + b_1i)(a_2 + b_2i) =$$

$$a_1a_2 + a_1b_2i + a_2b_1i + b_1b_2i^2 =$$

$$a_1a_2 + a_1b_2i + a_2b_1i - b_1b_2 =$$

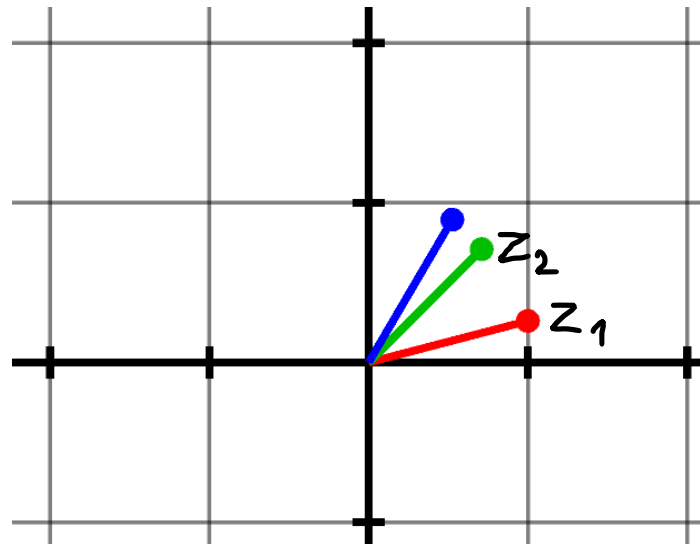
$$(a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$$

Operations

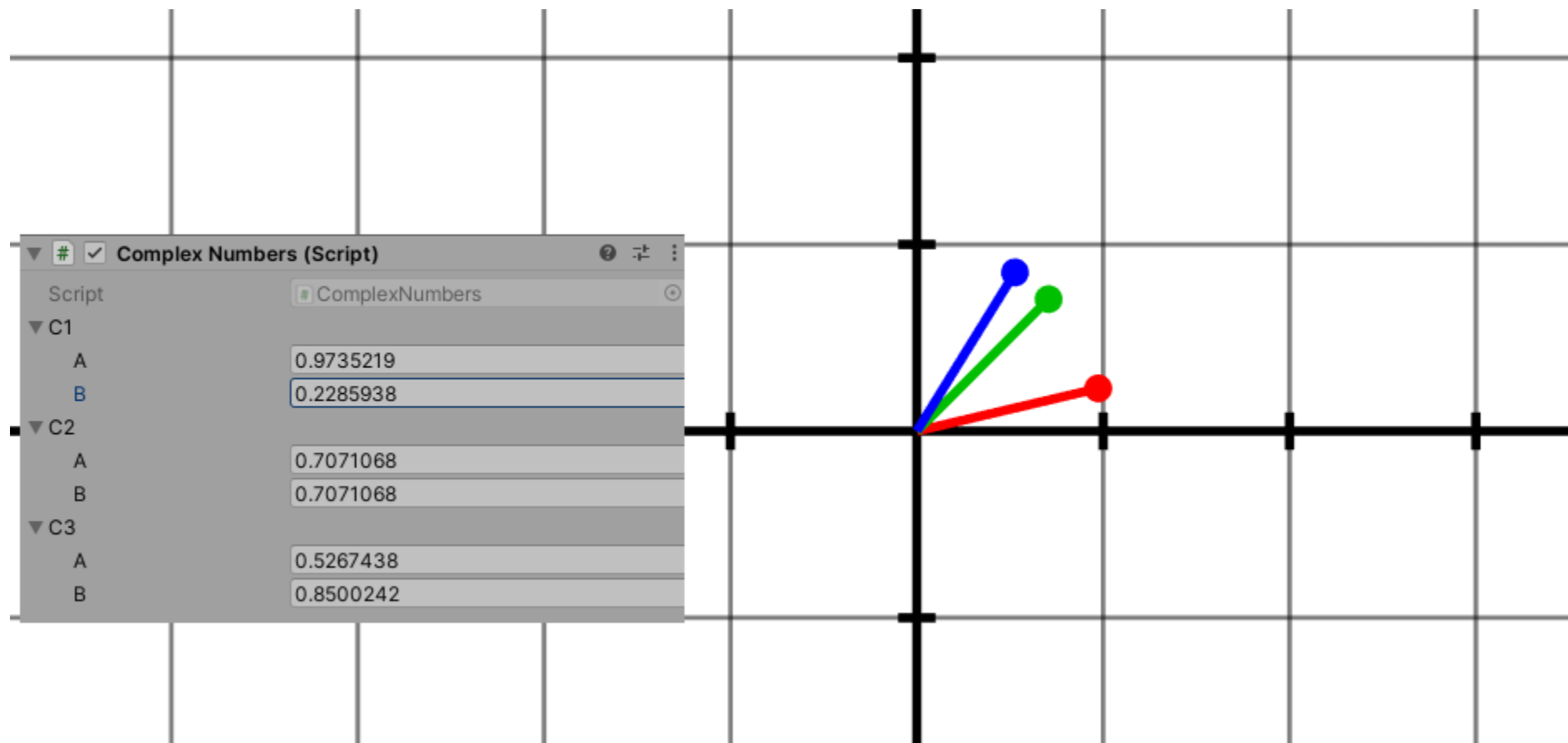
- Multiplying two complex numbers is **commutative**:

$$z_1 z_2 = z_2 z_1$$

- Multiplication of complex numbers has an interesting geometric interpretation:



Operations



Operations

- For any complex number we can calculate its **conjugate**:

$$z' = a - bi$$

- The **inverse** of a complex number is calculated like so:

$$z^{-1} = \frac{z'}{|z|^2}$$

- If the modulus of a complex number is 1, then:

$$z^{-1} = z'$$

Operations

- Let's prove the formula for the inverse assuming the modulus is 1:

$$z = a + bi$$

$$z^{-1} = z' = a - bi$$

$$z * z^{-1} = 1$$

$$(a + bi) * (a - bi) = a^2 - b^2i^2 = a^2 + b^2 = 1$$

Trigonometric Form

- A complex number can be represented using the **trigonometric form**:

$$x = a = r \cos(\varphi)$$

$$y = b = r \sin(\varphi)$$

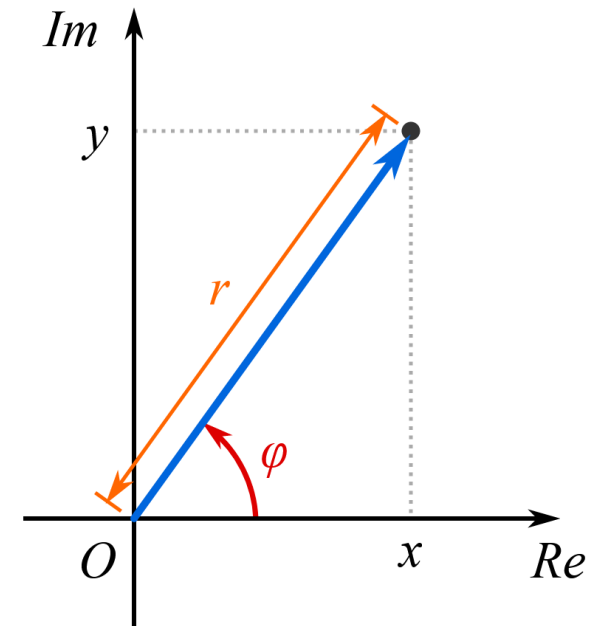
$$z = a + bi = r \cos(\varphi) + r \sin(\varphi) i$$

$$z = a + bi = r(\cos(\varphi) + i \sin(\varphi))$$

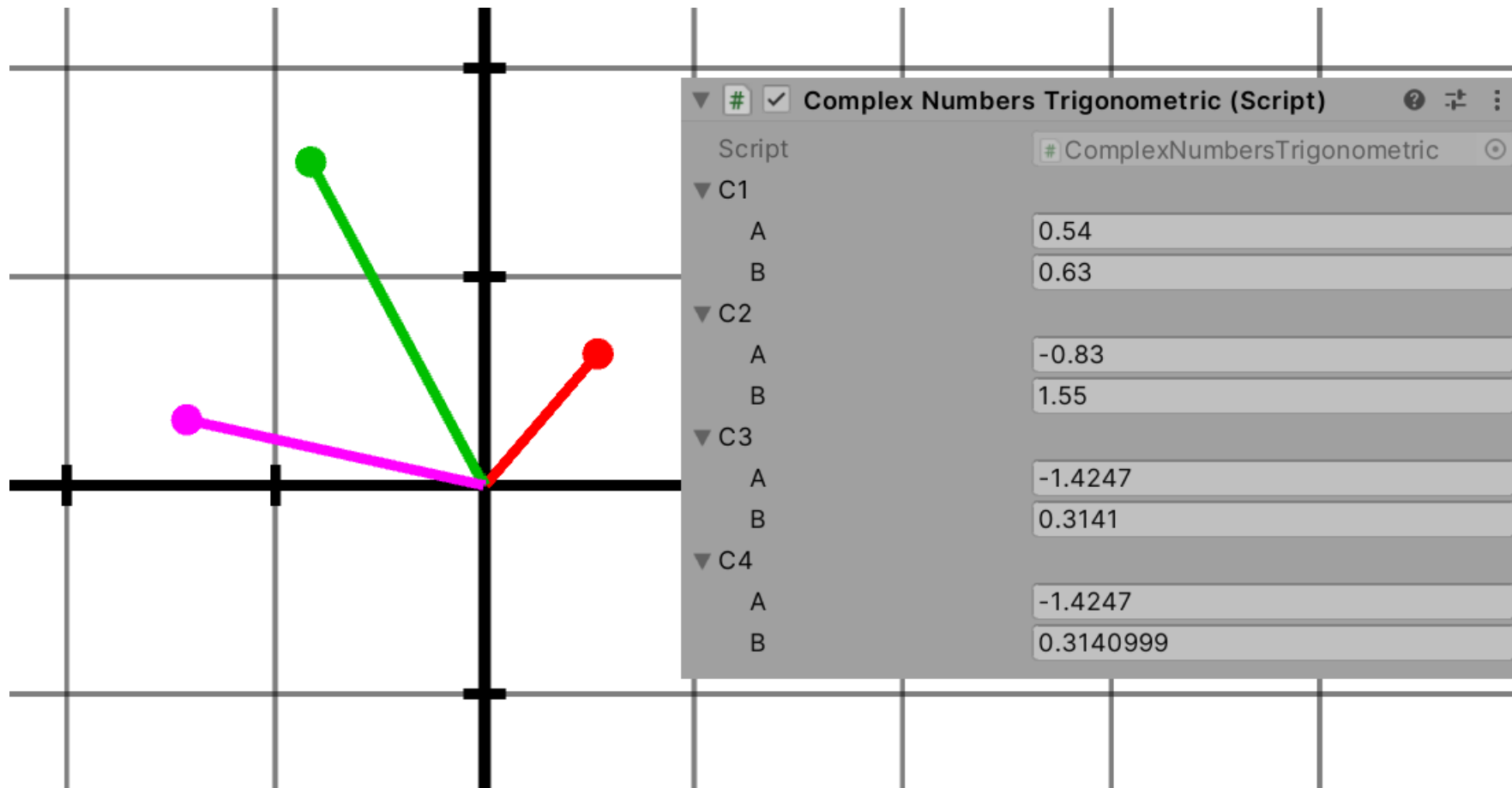
- It's like converting to polar coordinates:

$$\varphi = \tan^{-1} \left(\frac{y}{x} \right)$$

$$r = \sqrt{x^2 + y^2}$$



Trigonometric Form



Exponential Form

- Reminder: the algebraic form: $z = a + bi$
- Reminder: the trigonometric form: $z = r(\cos(\alpha) + i \sin(\alpha))$

- **Euler's formula:**

$$e^{i\alpha} = \cos(\alpha) + i \sin(\alpha)$$

- **Exponential form:**

$$z = r e^{i\alpha}$$

- It is worth knowing these three forms of writing a complex number, because different formulas related to them, in different fields and contexts, use different forms

Exponential Form

- Multiplying two complex numbers:

$$z_1 = r_1 e^{i\alpha_1}$$

$$z_2 = r_2 e^{i\alpha_2}$$

$$z_1 z_2 = (r_1 e^{i\alpha_1})(r_2 e^{i\alpha_2}) = r_1 r_2 e^{(i\alpha_1 + i\alpha_2)} = r_1 r_2 e^{i(\alpha_1 + \alpha_2)}$$

- When multiplying two complex numbers, their moduli ("lengths") are multiplied, and the angles add

Exercises

1. Calculate (slide 11):

$$(2 - 3i)(-3 + 9i)$$

2. Prove the following formula for the inverse of a complex number (slide 14):

$$z^{-1} = \frac{z'}{|z|^2}$$

Reminder: the inverse number is one for which:

$$z * z^{-1} = 1$$

Exercises

3. In program `ComplexNumbersTrigonometric`, in class `Complex`, store the complex number in the trigonometric form instead of the algebraic form (slide 17).
Keep the interface of the class intact