Math for 3D/Games Programmers

2. Complex Numbers

ATTENTION

- Complex numbers are not particularly common in 3D/games programming, so we will only skim over them
- The main reason we are touching on this subject is because complex numbers are foundation for **quaternions**, which in turn are widely used in 3D/games programming, and which will be discussed in chapter 7
- Complex numbers will sometimes show up (indirectly) when solving more complicated equations, what we will witness in chapter 8
- Moreover, complex numbers are important in other domains (e.g. Fourier transform) and it is worthwhile to have some basic understanding about them

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Definition

- **Complex number** is a number of the following form:

$$z = a + bi \qquad a, b \in R$$

$$i \ - \text{imaginary unit, such that:}$$
$$i = \sqrt{-1} \qquad \text{or} \qquad i^2 = -1$$

- Complex number is therefore represented by a pair of real numbers (a, b)
- Above we're seeing the **algebraic form** of a complex number
- Just as on the number line we can mark real numbers, we can mark complex numbers in the 2D coordinate system

Definition

- A few examples of complex numbers:

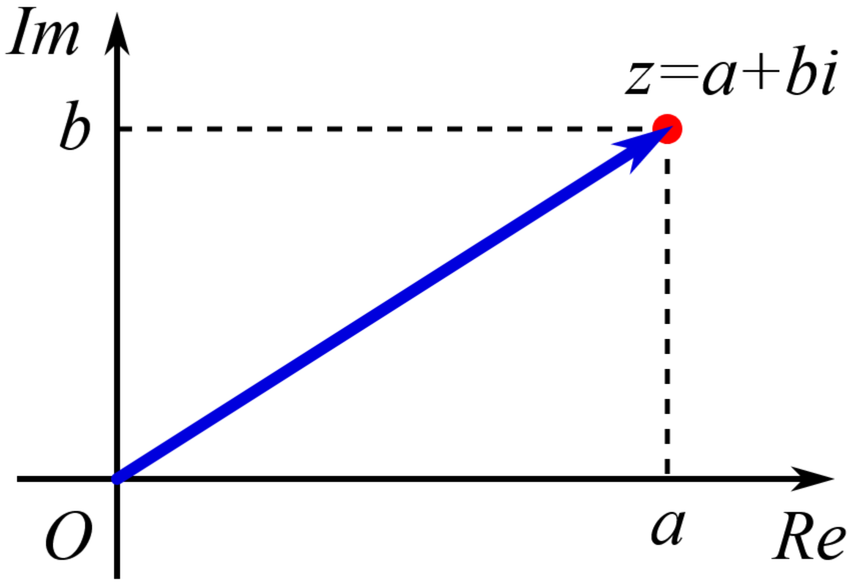
$$6 + 2i$$

$$-4 - \sqrt{3}i$$

$$-7i$$

$$5$$

Definition



https://en.wikipedia.org/wiki/File:A_plus_bi.svg Author IkamusumeFan

- Length of the blue arrow (distance of the z number from the origin) is the so called **modulus** („length”) of a complex number, which is calculated as:

$$|z| = r = \sqrt{a^2 + b^2}$$

Definition

- Note that since it is true that $i = \sqrt{-1}$, then in the complex numbers domain we can calculate roots of negative numbers, for example:

$$\sqrt{-4} \;=\; \sqrt{4} * \sqrt{-1} \;=\; 2\sqrt{-1} \;=\; 2i$$

- Wolfram
- The set of natural numbers $\mathbb{N}$ is contained in the set of integer numbers $\mathbb{Z}$, while the set of real numbers $\mathbb{R}$ is contained in the set of complex numbers $\mathbb{C}$:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Definition

- It happens that when solving some equations, the problem may be formulated in real numbers $\mathbb{R}$, the solution may also be in real numbers $\mathbb{R}$, but transformations of equations along the way may include complex numbers $\mathbb{C}$
- As an illustration let's solve some simple quadratic equation:

$$x^2 - 2x - 3 = 0$$

$$\Delta = b^2 - 4ac = 16 \qquad \sqrt{\Delta} = 4$$

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = -1 \mid 3$$

We started and finished with integer numbers $\mathbb{Z}$, but along the way we had to deal with fractions (rational numbers $\mathbb{Q}$)

Operations

- Many operations are defined on complex numbers, including operations that apply to real numbers, such as addition, multiplication, square root, etc.
- However, you cannot compare two complex numbers!
- For example, adding two complex numbers looks like this:

$$z_1 = a_1 + b_1 i$$
$$z_2 = a_2 + b_2 i$$

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$$

Operations

- Multiplying two complex numbers:

$$z_1 = a_1 + b_1 i$$
$$z_2 = a_2 + b_2 i$$

$$z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) =$$

$$a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2 =$$

$$a_1 a_2 + a_1 b_2 i + a_2 b_1 i - b_1 b_2 =$$

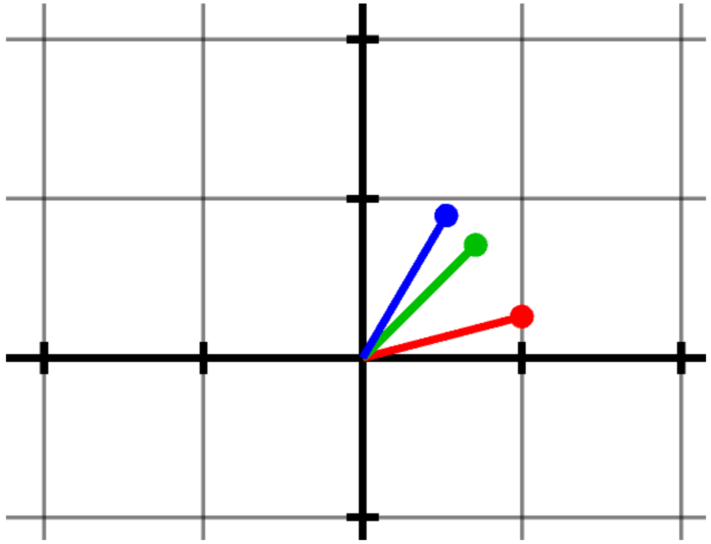
$$(a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i$$

Operations

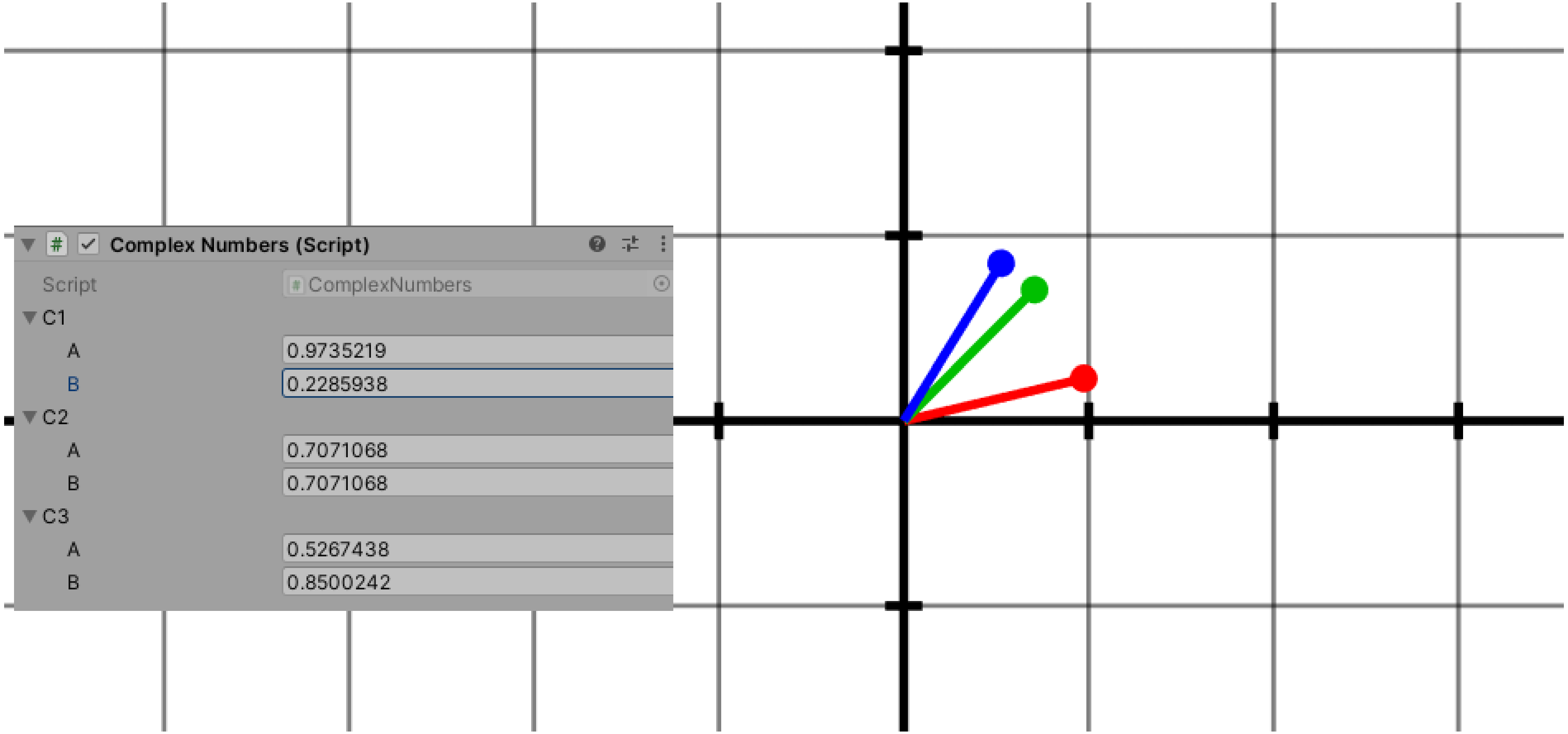
- Multiplying two complex numbers is **commutative**:

$$z_1 z_2 = z_2 z_1$$

- Multiplication of complex numbers has an interesting geometric interpretation:



Operations


Complex Numbers (Script)
Script
ComplexNumbers
C1
A
0.9735219
B
0.2285938
C2
A
0.7071068
B
0.7071068
C3
A
0.5267438
B
0.8500242

Operations

- For any complex number we can calculate its **conjugate**:

$$z' = a - bi$$

- The **inverse** of a complex number is calculated like so:

$$z^{-1} = \frac{z'}{|z|^2}$$

- If the modulus of a complex number is 1, then:

$$z^{-1} = z'$$

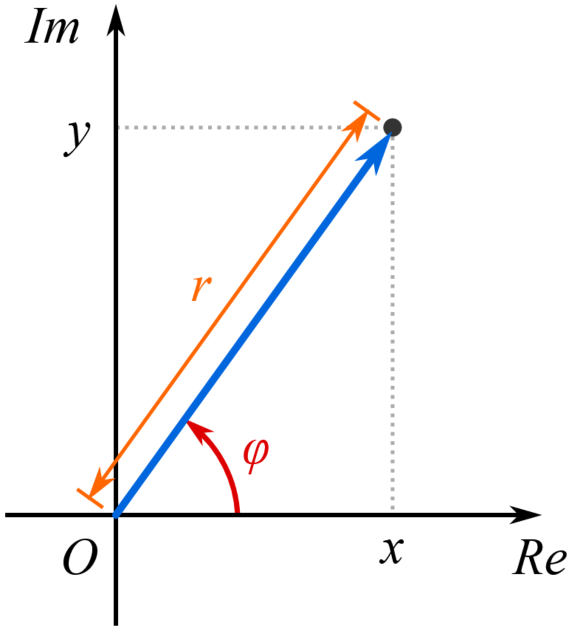
Trigonometric Form

- A complex number can be represented using the **trigonometric form**:

$$
\begin{aligned}
x &= a = r\cos(\alpha) \\
y &= b = r\sin(\alpha)
\end{aligned}
$$

$$
\begin{aligned}
z &= a + bi = r\cos(\alpha) + r\sin(\alpha)\, i \\
z &= a + bi = r(\cos(\alpha) + i\sin(\alpha))
\end{aligned}
$$

- It's like converting to polar coordinates



https://en.wikipedia.org/wiki/File:Complex_number_illustration_modarg.svg **Author** Kan8eDie

Exponential Form

- Reminder: the algebraic form: $z = a + bi$
- Reminder: the trigonometric form: $z = r(\cos(\alpha) + i\sin(\alpha))$
- **Euler's formula**:

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$$

- **Exponential form**:

$$z = re^{i\alpha}$$

- It is worth knowing these three forms of writing a complex number, because different formulas related to them, in different fields and contexts, use different of these forms

Exponential Form

- Multiplying two complex numbers:

$$z_1 = r_1 e^{i\alpha_1}$$

$$z_2 = r_2 e^{i\alpha_2}$$

$$z_1 z_2 \;=\; \left(r_1 e^{i\alpha_1}\right)\left(r_2 e^{i\alpha_2}\right) \;=\; r_1 r_2 e^{(i\alpha_1 + i\alpha_2)} \;=\; r_1 r_2 e^{i(\alpha_1 + \alpha_2)}$$

- When multiplying two complex numbers, their moduli ("lengths") are multiplied, and the angles add

Exercises

1. Calculate (slide 10):

$$(2-3i)(-3+9i)$$

2. Prove the following formula for the inverse of a complex number:

$$z^{-1} = \frac{z'}{|z|^2}$$

Reminder: the inverse number is one for which:

$$z * z^{-1} = 1$$